ENERGY MOMENTUM TENSOR IN THEORIES WITH SCALARS, BEHAVIOR NEAR FIXED POINT AND APPLICATIONS TO GRAVITATIONAL SCATTERING

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ENERGY MOMENTUM TENSOR IN THEORIES WITH SCALARS, BEHAVIOR NEAR FIXED POINT AND APPLICATIONS TO GRAVITATIONAL SCATTERING

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CERTIFICATE

Certified that the work contained in this thesis entitled "Energy momentum tensor in theories with scalars, behavior near fixed point and applications to gravitational scattering" has been carried out by Ms. Achla Mishra under my supervision and has not been submitted elsewhere for a degree.

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- Achla Mishra

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"ENERGY MOMENTUM TENSOR IN THEORIES WITH SCALARS, BEHAVIOR NEAR FIXED POINT AND APPLICATIONS TO GRAVITATIONAL SCATTERING"

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Renormalization of energy momentum tensor (that couples to the gravity) in theories involving scalars, has been studied extensively. The matrix elements of energy momentum tensor in these theories can always be made finite by the introduction of infinite counterterm but in this case there is a need for the extra renormalization condition involving experimental input and anew parameter (apart from the flat space parameters) to describe the theory in the presence of gravity. But if the energy momentum tensor is made finite using finite improvement program, then no independent renormalization is needed to renormalize the energy momentum tensor and hence no extra parameter from the experiment is required.

Collins has shown that finite improvement program goes through in $\lambda\phi^4$ theory. Finite improvement program fails in theories with scalars and two coupling constants. The failure depended crucially upon the fact that there were two independent coupling constants present. In a part of this thesis we have worked on the finite improvement program in scalar QED at $\lambda = 0$, which is a theory with only one independent coupling constant e_o. We have found that the finite improvement program in this theory works to a high order in coupling constant i.e. upto $O(e^{10})$ but necessarily fails in $O(e^{14})$ and most probably in $O(e^{12})$ itself. Our results also help us to understand the failure of finite improvement program in theories with scalars and two coupling constants. It appears that $\lambda\phi^4$ theory is the exceptional case where finite improvement program works. In theories involving scalars and two coupling constants, one necessarily has to introduce an infinite counterterm to obtain energy momentum tensor that is finite to all orders in perturbation theory but extra experimental information and new parameter seem to be needed.

In this thesis we have worked further on an independent <u>theoretical</u> criterion known as 'renormalization group covariance' criterion suggested by Collins first, in the context of $\lambda\phi^4$ theory, to fix the energy momentum tensor uniquely in scalar QED and Weinberg-Salam model. We have shown that in these theories it is possible to obtain a unique, finite energy momentum tensor when 'renormalization group covariance' condition together with certain boundary condition are used. The advantage of fixing the energy

momentum tensor using this criterion is that then no experimental input is required though the energy momentum tensor does have infinite renormalization. In view of the fact that the finite improvement program fails in theories with scalars and two coupling constants, our study becomes important.

We have also worked on the behavior of the trace anomaly of Collins energy momentum tensor in $\lambda\phi^4$ theory, near the fixed point of the theory. It has been argued that for the massless case, this trace anomaly vanishes near the fixed point. We have taken a closer scrutiny of this claim and found that in general when the theory has a non trivial fixed point, this may not be true. The trace anomaly instead of vanishing may infact blow up near the fixed point. We have shown that the Green's functions with the insertion of energy momentum tensor discovered by Collins can blow up near the fixed point, if the anomalous dimension associated with mass, $\gamma_m(\lambda^*) > 0$. It has been shown that these Green's functions scale with an extra anomalous dimension $2\gamma_m(\lambda^*)$ near $\lambda = \lambda^*$ leading to a non trivial high energy behavior near the fixed point. The energy momentum tensor in scalar QED and Weinberg-Salarm model also may blow up near the fixed point of the theory, thus leading to a similar high energy behavior of its Green's functions.

In the last part of the thesis, we have applied the results mentioned above to few physical processes in $\lambda\phi^4$ theory. The processes are the scattering of a scalar by an external

gravitational field and the scattering of two scalars in presence of external gravitational field. We have found that the differential scattering crosssections associated with these processes, can rise rapidly with energy, if $\gamma_m(\lambda^*)$ is large enough. The same results are expected to hold for Weinberg-Salam model also. Even if $\lambda\phi^4$ theory does not have a non trivial fixed point, our results which can be generalized to Weinberg-Salam model have a physical relevance, if the latter theory has a non trivial fixed point.

CHAPTER - 1

INTRODUCTION

Energy momentum tensors in quantum field theories have been studied extensively [1-12]. At the classical level, the energy momentum tensor $\theta_{\mu\nu}^{\text{canonical}}$ is defined as the current associated with the space time translations, and is a conserved quantity. $\int\theta_{\mu o}^{\text{can}}\, d^3x$ gives the four momentum of the system. Energy momentum tensor $\theta_{\mu\nu}^{\text{can}}$ enters in the classical definitions of canonical dilatation and conformal currents. But it is only when we consider the gravitational interactions that the full physical significance of energy momentum tensor emerges. In the development of the quantized theory of gravitational interactions, there are two stages. The first stage is to extend the quantum field theory describing the weak, electromagnetic or strong processes, to incorporate an external gravitational field and in the second stage the gravitational field is quantized and the interaction of gravitation and matter fields is considered. The study of the first stage of the theory involves the Green's functions with the insertion of energy momentum tensor, since in a weak external gravitational field $h_{\mu\nu}$, it is the energy momentum tensor of the matter fields determined by the general covariance considerations (which we call Einstein tensor) defined by,

$$\theta_{\mu\nu}(y) = \frac{2}{\sqrt{-g(y)}} \frac{\delta S}{\delta g^{\mu\nu}(y)}\bigg|_{g^{\mu\nu}(y) = \eta^{\mu\nu}} \qquad \ldots.(1.1)$$

which couples with the gravity. The knowledge of the matrix elements of $\theta_{\mu\nu}$ is needed to describe the scattering in a weak external gravitational field. These matrix elements (when the external lines are put on mass shell) are observable and hence should be finite. Besides this, in practical experiments in particle physics there is always a weak external gravitational field present, and it would be quite uncomfortable from a theoretical point of view if its effect were calculated and found to be infinite instead of small. Hence we desire $\theta_{\mu\nu}$ to be finite.

We shall describe the renormalization of Einstein tensor $\theta_{\mu\nu}$, in $\lambda\phi^4$ theory, to start with, since it is the simplest renormalizable quantum field theory, described by the action

$$S = \int d^4x \; [\tfrac{1}{2} \partial_\mu\phi\partial^\mu\phi - \frac{m_o^2}{2} \phi^2 - \frac{\lambda_o}{4!} \phi^4] \;\; ; \;\; S = \int d^4x \; \mathcal{L}$$

To obtain $\theta_{\mu\nu}$, we change the flat space action, written above, to the curved space action which is generally covariant and is obtained in the minimal way by changing the ordinary derivatives to covariant derivatives, $\int d^4x \to \int\sqrt{-g}\, d^4x$ and $\eta^{\mu\nu} \to g^{\mu\nu}(x)$. It is given by

$$S[\phi,g] = \int d^4x \sqrt{-g(x)} \; [\frac{g^{\mu\nu}}{2} \partial_\mu\phi \; \partial_\nu\phi - \frac{m_o^2}{2} \phi^2 - \frac{\lambda_o}{4!} \phi^4]$$

The energy momentum tensor obtained using equation (1.1) is written below:

$$\theta_{\mu\nu} = \partial_\mu\phi \; \partial_\nu\phi - \eta_{\mu\nu} \mathcal{L}$$

which is the same as the canonical energy momentum tensor in $\lambda\phi^4$ theory. This $\theta_{\mu\nu}$ does not have finite matrix elements even to the lowest order in λ. But we can always add to it certain term to make it finite, provided it does not affect the conserved quantities $\int \theta_{\mu o} d^3x$. In fact, the most general energy momentum tensor which we shall denote by $\theta_{\mu\nu}^{imp}$, which satisfies the following properties:

i) $\partial^\mu \theta_{\mu\nu}^{imp}$ vanishes, when the classical equations of motion are used.

ii) $\int \theta_{\mu o}^{imp} d^3x$ gives the four momentum of the system correctly.

iii) It contains terms of dimension n or less, is given by

$$\theta_{\mu\nu}^{imp} = \partial_\mu\phi\ \partial_\nu\phi - \eta_{\mu\nu}\mathcal{L} + H_o(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2 \qquad (1.2)$$

$\theta_{\mu\nu}^{imp}$ can be derived from the non minimal generally covariant action

$$S = \int d^4x\ \sqrt{-g(x)}\ (\mathcal{L}_{min} - \frac{H_o}{2} R\phi^2)$$

where R is the curvature scalar. Callan, Coleman and Jackiw [2] were the first to show that if one choses $H_o = -\frac{1}{6}$, the energy momentum tensor is finite to one loop order. Freedman, Muzinich and Weinberg [4] showed that this improved energy momentum tensor has finite matrix elements at zero external momentum q, and to first order in q, but for $q \neq 0$, the finiteness for general q was

proven only to one loop order. The same result was shown for the renormalizable gauge field theories with scalar mesons and the matrix elements of $\theta^{imp}_{\mu\nu}$ were shown to be gauge independent. Freedman and Weinberg [5] considered the problem in $\lambda\phi^4$ theory in the context of dimensional regularization and showed that a proper choice of $H_o = H_o(\lambda)$ can be made to $O(\lambda^2)$, where $H_o(\lambda)$ is a power series in λ with finite coefficients, leading to a finite energy momentum tensor to two loop level but it failed beyond two loops. It lead them to conclude that it was not possible to make $\theta_{\mu\nu}$ finite in $\lambda\phi^4$ theory, unless new infinite counterterms, beyond those required to renormalize Green's functions of the flat space theory, were introduced. In such a case, the term $\frac{R\phi^2}{12}$ is replaced by $\frac{Z_g}{12}(1+g)\,R\phi^2$, where a new parameter g has been introduced to remove the arbitrariness in the renormalization procedure. The infinite counterterm Z_g is determined order by order so that the renormalization condition

$$\frac{d}{dq^2}\,\Gamma^{(2)}_{\theta^{\mu}_{\mu}}\left(q;\,p,\,-p-q\right)\Big|_{q^2=0} = -g \quad \text{is satisfied.}$$

There is no reason to prefer a particular value of g and is fixed by the experiment. In the same work, it was also shown that in renormalizable gauge theories with scalar mesons, it is possible to describe the first order interactions with external gravitational field by an energy momentum tensor which has finite matrix elements but we must introduce a new infinite renormalization constant and a new parameter which must be

determined experimentally. (As a side remark, we wish to point out that in gauge theories involving fermions but no scalar, the Einstein energy momentum tensor is finite and does not need any improvement term [4]). The introduction of infinite counterterm to renormalize $\theta_{\mu\nu}$ implies that the coefficient H_o is renormalized independently. Collins [8]considered the problem of renormalization of energy momentum tensor $\theta_{\mu\nu}$ in $\lambda\phi^4$ theory in generality. He showed that there is a unique $H_o\left(\lambda, \frac{m^2}{\mu^2}, \epsilon\right)$, where H_o is the finite function of renormalized quantities at $\epsilon = 0$, which gives finite matrix elements for the energy momentum tensor to all orders in perturbation theory. This improvement coefficient is a function of ϵ only and is finite at $\epsilon = 0$. It is a power series in ϵ and the coefficients of ϵ^r can be chosen appropriately in each order in perturbation theory so as to make the matrix elements of $\theta_{\mu\nu}$ finite to all orders. $H_o\left(\lambda, \frac{m^2}{\mu^2}, \epsilon\right)$ is not a finite function of bare quantities (except in the special case when it is a function ϵ only). Thus such an energy momentum tensor is generally not derivable from an action that is a finite function of bare quantities. In general, total actions (including renormalization counterterms) are finite functions of bare quantities. Consequently the energy momentum tensor derived from them, are finite function of bare quantities. For example, the energy momentum tensor in gauge theories without scalars Ref. [4] is a finite function of bare quantities. Thus it is more desirable to obtain an energy momentum tensor in scalar theories as well, which is finite function of bare quantities and has

finite elements to all orders in perturbation theory. Such an energy momentum tensor

$$\theta^{imp}_{\mu\nu} = \theta_{\mu\nu} + H_o\left[\lambda_o\mu^{-\epsilon},\ \frac{m_o^2}{\mu^2},\ \epsilon\right]\ (\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2 \qquad (1.3)$$

where H_o is a finite function of its arguments $\lambda_o\mu^{-\epsilon}$ and $\frac{m_o^2}{\mu^2}$ at $\epsilon = 0$, can then be derived from an action

$$S[g,\phi] = \int d^nx\ \sqrt{-g}\ \mathcal{L} + \frac{H_o}{2}\left[\lambda_o\mu^{-\epsilon},\ \frac{m_o^2}{\mu^2},\ \epsilon\right] \int\sqrt{-g}\ R\phi^2\ d^nx$$

that is a finite function of bare quantities. Energy momentum tensor of the kind given by equation (1.3) exists since Collins energy momentum tensor

$$\theta^{imp}_{\mu\nu} = \theta_{\mu\nu} + H_o(\epsilon)\left(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2\right)\phi^2 \qquad (1.4)$$

is a finite energy momentum tensor of the form of equation (1.3). In ref. [13], it has been shown that the energy momentum tensor given by equation (1.4) is a unique energy momentum tensor of the form of equation (1.3), that is finite to all orders in perturbation theory. The schemes of obtaining finite energy momentum tensor using either kinds of Ho's are called finite improvement programs. More on this will be found in the next chapter. The finite improvement program has been tried in the theories with scalars and two coupling constants. It has been shown that it does not work in these theories [14,15]. Thus $\lambda\phi^4$ theory appears to be an exceptional case where it works. Brown

[11] gave an alternate construction for a finite energy momentum tensor in $\lambda\phi^4$ theory, but this energy momentum tensor has the property that the improvement term is non trivially renormalized in higher orders and contains negative powers of ϵ. This should be contrasted with $\theta_{\mu\nu}$ with improvement term of the kind $H_0(\epsilon)$ $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial\ )\phi^2$, which needs no counterterms.

In the present thesis, we have worked on the renormalization of energy momentum tensor and its behavior near the fixed point in different renormalizable quantum field theories. The plan of the thesis is as follows. In the second chapter, we have dealt with the finite improvement program in scalar QED at $\lambda = 0$. The analysis carried out there sheds some light on the failure of finite improvement program in theories with scalars and two coupling constants. The work of the next two chapters is related to obtaining a unique, finite energy momentum tensor in scalar QED and Weinberg-Salam model. Freedman and Weinberg had found that the energy momentum tensor (which couples to the gravity) can be renormalized by adding an infinite counterterm to it but an extra renormalization condition involving a new parameter, to be fixed by the experiment is needed. We have used a theoretical criterion suggested by Collins known as `Renormalization Group Covariance'criterion for fixing the finite energy momentum tensor uniquely in these theories. In this case, no new information from the experiments is required though the energy momentum tensor does have infinite renormalization. In chapter 5. We have dealt with the behavior of trace anomaly and energy momentum tensor

discovered by Collins, near the fixed point of $\lambda\phi^4$ theory We have also discussed the behavior of energy momentum tensor in scalar QED and W-S model near the fixed points of the theories. We have shown that in these theories, the Green's functions with the insertion of energy momentum tensor, can show a non trivial behavior near the ultraviolet fixed point of the theory, provided certain conditions are met. In chapter 6, we have applied these results to the case of the scattering of a scalar by an external gravity and to the gravitational corrections to the scattering of two scalars in $\lambda\phi^4$ theory, and obtained the high energy behavior of the scattering crosssections for these processes. It is shown that for the scattering a scalar in an external gravity, the cross-sections can rise with energy if the anomalous dimension associated with mass, $\gamma_m(\lambda^*)$ is large enough. In the context of scattering of two scalars, it is shown that the differential scattering crosssection has a piece that can, under certain conditions, rise with energy. Also the differential scattering crosssection in the presence of gravity can become comparable with the differential scattering crosssection in the presence of strong interaction, at energy scale of say 25 TeV [27].

CHAPTER - 2

RENORMALIZATION OF ENERGY MOMENTUM TENSOR IN SCALAR QED

2.1 Introduction

The significance of Einstein energy momentum tensor $\theta_{\mu\nu}$ in quantum field theories has been dealt with, inthe previous chapter. We expect the energy momentum tensor that couples to gravity to have finite matrix elements since its matrix elements are observables. Moreover the energy momentum tensor appears in the operator product expansion (O.P.E.) of two currrents. This O.P.E. is related to physical scattering matrix elements and therefore the finiteness (or the lack of finiteness) and its anomalous dimension have observable consequences. We considered some examples of renormalizable field theories, in which in order to obtain the finite matrix elements of the energy momentum tensor, it is necessary to add to it certain `improvement terms' allowed by the general covariance. In this work, our focus will be on the renormalization of energy momentum tensor $\theta_{\mu\nu}(x)$ in scalar QED, in which $(\phi^*\phi)^2$ coupling is induced only through electromagnetic interactions. We wish to work with this special case of scalar QED for reasons which wil be discussed shortly.

The energy momentum tensor $\theta_{\mu\nu}$ obtained from the action is given by

$$\theta_{\mu\nu}(x) = \frac{2}{\sqrt{-g(x)}} \left. \frac{\delta S}{\delta g^{\mu\nu}(x)} \right|_{g^{\mu\nu}(x) = \eta^{\mu\nu}} \tag{2.1}$$

where S is the curved space action which is obtained from the flat space action of scalar QED theoryby the following substitutions: all ordinary derivatives ⟶ covariant derivatives, $\int d^n x \to \int \sqrt{-g}\, d^n x$ and $\eta_{\mu\nu} \to g_{\mu\nu}(x)$. This energy momentum tensor is not finite even in the one loop order and an improvement term is necessarily needed. In scalar QED, the divergences that are present in the Green's functions with the insertion of $\theta_{\mu\nu}(x)$ are of the kind $(\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial^2)\ (\phi^* \phi)$ as shown in Ref. [4,5]. Hence the first step towards improving $\theta_{\mu\nu}(x)$ in scalar QED so as to get its finite matrix elements (atleast to some loop order if not to all orders) would be to try adding CCJ improvement term as done in $\lambda\phi^4$ theory. If we wish to derive the improved energy momentum tensor from the action S, we must add to it, a non minimal term of the kind $\frac{H_o}{2}\int R\phi^*\phi\ d^n x$ [R=0 for the flat space but its derivative with respect to $g_{\mu\nu}(x)$ gives the improvement term]. The improved $\theta_{\mu\nu}(x)$ derived from the action S (which includes the CCJ improvement term generalized to dimensional regularization with $H_o = \frac{(n-2)}{8(1-n)}$) is given by

$$\theta_{\mu\nu}^{imp} = \theta_{\mu\nu} + \frac{(n-2)}{4(1-n)}\ (\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial^2)\ (\phi^* \phi) \qquad (2.2)$$

This $\theta_{\mu\nu}$ is finite upto $O(\lambda^3)$ at e=0, upto $O(e^4)$ at λ=0 and also in $O(\lambda e^2)$, but a further improvement is needed in $O(\lambda^4)$, $O(\lambda^2 e^2)$ and $O(\lambda e^4)$[14]. $\theta_{\mu\nu}$ can always be made finite by adding an infinite counterterm to it, where H_o will now contain the required counterterm. But such an energy momentum tensor has inverse powers of ε in its definition and leads to infinite renormalization of $\theta_{\mu\nu}$. This counterterm is not present in the

original conception of the theory. As pointed out in the first chapter, to remove the arbitrariness in the renormalization procedure, an extra renormalization condition involving experimental input and a new parameter (not present in the flat space theory) is required. But the need for this new parameter to renormalize $\theta_{\mu\nu}$ violates the spirit of principle of general covariance which suggests that the flat space parameters be sufficient to describe the theory in the presence of gravity. Hence this is the drawback of renormalizing $\theta_{\mu\nu}$ by adding an infinite counterterm to it. Extra experimental input is not needed if it is possible to renormalize $\theta_{\mu\nu}$ using `finite improvement program', henceforth called FIP. Also in this case, the flat space parameters are sufficient to describe the theory completely in the presence of external gravity. We shall explain these points now. The FIP is of two kinds I and II. FIP I is one, in which the coefficient H_o of the improvement term $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)(\phi^*\phi)$ in $\theta_{\mu\nu}^{imp}$ is the finite function of unrenormalized parameters of the theory at $\epsilon=0$. And in FIP II, the coefficient H_o is a finite function of renormalized quantities at $\epsilon = 0$. Now, if H_o is a finite function of the bare quantities at $\epsilon = 0$, no new counterterms are needed which would correspond to the renormalization of H_o (apart from the flat space renormalizations). Whereas, when H_o is a finite function of renormalized quantities at $\epsilon=0$, the renormalization counterterms are present but no extra renormalization condition is required to fix them. For example, in the context of $\lambda\phi^4$ theory, we can always write

$$H_o(\lambda, \frac{m^2}{\mu^2}, \varepsilon) = H'_o(\lambda_o \mu^{-\varepsilon}, \frac{m_o^2}{\mu^2}, \varepsilon) + Z_{H_o}(\lambda_o \mu^{-\varepsilon}, \frac{m_o^2}{\mu^2}, \frac{1}{\varepsilon})$$

$H'_o(\lambda_o \mu^{-\varepsilon}, \frac{m_o^2}{\mu^2}, \varepsilon)$ is a finite function of bare quantities and Z_{H_o} is the infinite function of bare quantities. Hence the counterterms are present in FIP II but they get fixed by the requirement that on the left hand side of the above expression, H_o is a finite function of renormalized quantities at $\in = 0$ H_o does not get renormalized independently. In both kinds of FIP, $\theta_{\mu\nu}$ is renormalized without requiring experimental input and a new parameter (not present in the flat space theory). Since the renormalizability of Einstein tensor gives the renormalizability of the combined theory of gravity and matter, with gravity treated to lowest order and the self interactions of matter to all orders [6], so if it is possible to make $\theta_{\mu\nu}$ finite using FIP, the flat space parameters are sufficient to describe the theory in the presence of external gravity. Therefore, it is more desirable to try FIP to renormalize $\theta_{\mu\nu}$ in any renormalizable field theory.

As shown by Collins [8], the FIP of both the kinds I and II work in $\lambda\phi^4$ theory. In this theory the improved $\theta_{\mu\nu}$ of the kind

$\theta_{\mu\nu}^{imp} = \theta_{\mu\nu} + H_o(\in)\ (\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2$ is finite to all orders in perturbation theory. $H_o(\in) = \sum_{p=0}^{\infty} h_p \in^p$ is a perturbative series in non negative powers of $\in = 4-n$, where n is the space time dimension. FIP works because it is possible to choose h_p's uniquely in each order in perturbation theory such that improved

$\theta_{\mu\nu}$ is finite. This point has been explained in chapter 5 also. The success of FIP in $\lambda\phi^4$ theory motivated authors [13,14] to apply it to scalar QED and other renormalizable field theories with scalar fields and two coupling constants. They found that in the theories with scalar fields and two coupling constants, the FIP of either kinds do not go through. In the context of scalar QED, they showed that even in $O(e^2)$ it is not possible to choose either kinds of H_0 such that $\theta^{imp}_{\mu\nu}$ is finite to all orders in λ. The proofs depended crucially upon the fact that two coupling constants were present. Hence one hopes that if there is a theory in which one has scalar fields but one coupling constant, the FIP might go through as it does in $\lambda\phi^4$ theory. In scalar QED, this can be arranged if one assumes that the $(\phi^*\phi)^2$ coupling is induced only through electromagnetic interactions. Here, we have investigated if the FIP goes through in such theory.

The importance of this work lies in the fact that though in ref.[14,15], it was shown that FIP does not go through in theories with scalar fields and two coupling constants butit did not tell us why it was so, whereas the analysis that we have carried out here, helps us to understand it in the special case of scalar QED at $\lambda=0$. We have shown that unlike in $\lambda\phi^4$ theory, in this theory, it is impossible to choose parameters in the improvement coefficient perturbatively in each order in coupling constant such that improved $\theta_{\mu\nu}$ is finite to all orders[16]. The above mentioned result is consistent with the fact that it is impossible to make the energy momentum tensor finite using finite improvement program in theories with scalars and two coupling

constants. Our work also indicates that $\lambda\phi^4$ theory is the exceptional case in which the finite improvement program works. Hence for the theories with scalars and more than one coupling constant, one would always need an extra renormalization condition involving the experimental input and a new parameter to renormalize $\theta_{\mu\nu}$.

2.2 Preliminaries

We shall work with a complex scalar field coupled to an abelian gauge field A_μ described by the Lagrange density

$$\mathcal{L}' = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} (D_\mu\phi)^*(D^\mu\phi) - \frac{1}{2} m_o^2 \phi^*\phi - \frac{1}{2} \xi_o(\partial.A)^2 \tag{2.3}$$

where $D_\mu\phi = \partial_\mu\phi - ie_oA_\mu\phi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

There is no $(\phi^*\phi)^2$ interaction in the lowest order, and thus $\mathcal{L}'$ contains only one independent coupling constant e_o. However, in $O(e_o^4)$ and higher, a $(\phi^*\phi)^2$ coupling is induced and there are counterterms needed of the form $\mu^\epsilon\, \delta\lambda(e^2,\epsilon)\int\frac{(\phi^*\phi)^2}{4!}\, d^nx$. Thus the original Lagrange density $\mathcal{L}'$ must be modified to contain a counterterm:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} (D_\mu\phi)^*(D^\mu\phi) - \frac{1}{2} m_o^2 \phi^*\phi$$
$$- \mu^\epsilon\, \delta\lambda(e^2,\epsilon) \frac{(\phi^*\phi)^2}{4!} - \frac{1}{2} \xi_o(\partial.A)^2 \tag{2.4}$$

This Lagrange density is a special case of the Lagrange

density

$$\mathcal{L}'' = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{1}{2} (D_\mu\phi)^*(D^\mu\phi) - \frac{1}{2} m_o^2 \phi^*\phi - \frac{\lambda_o}{4!}(\phi^*\phi)^2 - \frac{1}{2} \xi_o(\partial.A)^2 \tag{2.5}$$

$\int\mathcal{L}'' d^n x$ is an action containing two independent coupling constants λ_o and e_o and it is often convenient to look upon $\mathcal{L}$ of Eq. (2.4) as a limiting case of $\mathcal{L}''$ of Eq. (2.5), the limit being specified below. However, we should remark that in this work, we are interested in the Lagrange density $\mathcal{L}$ of Eq.(2.4) which has only <u>one</u> independent coupling constant e_o and that $\mathcal{L}''$ of Eq. (2.5) is being introduced only for certain technical reason . This reason is that, in general, the relation $\{O_i\}^{UR} = Z_{ij} \{O_j\}^R$, together with the following properties:

(i) $\langle O_i\rangle^{UR}$ are independent of μ.

(ii) $\langle O_j\rangle^R$ are finite and linearly independent.

(iii) Z_{ij} is the invertible renormalization matrix, gives the expression for the anomalous dimension γ_{ij} of the operator which is, $Z^{-1}_{ik} \mu\frac{\partial}{\partial\mu} Z_{kj} = \gamma_{ij}$ = finite at $\epsilon = 0$.

But if one starts with the lagrangian given by equation (2.4), the unrenormalized Green's functions with the insertion of operator O_i become μ dependent and the above derivation of γ_{ij} is not possible.

We shall use the dimensional regularization together with the minimal subtraction scheme for regularizing proper vertices of the theory and of operators. The renormalization transformations are

$$\phi = Z^{1/2}\phi^R , \quad m_o^{\,2} = m^2 Z_m , \quad e_o^{\,2} = \mu^{\varepsilon} e^2 Z_e^{\,2}$$

$$\lambda_o = \mu^{\varepsilon} \left[\lambda Z_\lambda (\lambda, e^2, \varepsilon) + \delta\lambda (e^2, \varepsilon)\right]$$

$$A_\mu = Z_3^{1/2} A_\mu^{\,R} , \quad \xi_o = Z_\xi \xi = Z_3^{-1} \xi \qquad (2.6)$$

Lagrange density of Eq. (2.4) is obtained from that of Eq. (2.5) by putting $\lambda=0$. So we shall deal with $\mathscr{L}''$ of Eq. (2.5) first, and obtain results for $\mathscr{L}$ of Eq. (2.4) by setting $\lambda=0$.

2.3 Conditions for a finite energy momentum tensor

We start with the following expression for the action:

$$S[A_\mu ,\phi, g] = \int d^n x \sqrt{-g}(x) \Big[- \frac{1}{4} g^{\alpha\beta}(x)\, g^{\gamma\delta}(x)\, F_{\alpha\gamma}\, F_{\beta\delta} +$$

$$\frac{1}{2} g^{\alpha\beta}(x)\, (D_\alpha\phi)^* (D_\beta\phi) - \frac{m_o^2}{2} \phi^*\phi - \frac{\lambda_o}{4!} (\phi^*\phi)^2$$

$$- \frac{\xi_o}{2} \left\{ \frac{1}{\sqrt{-g(x)}}\, \partial_\alpha \left\{ g^{\alpha\beta}(x)\sqrt{-g}(x)\, A_\beta \right\}\right\}^2$$

Einstein energy momentum tensor is defined as

$$\theta_{\mu\nu} = \frac{2}{\sqrt{-g(y)}} \quad \frac{\delta S}{\delta g^{\mu\nu}(y)} \Bigg|_{g^{\mu\nu}(y)= \eta^{\mu\nu}}$$

To prove the finiteness of $\theta_{\mu\nu}$ it is sufficient to prove the finiteness of its trace [4]. The trace of the above $\theta_{\mu\nu}$ is

$$\theta^\mu_\mu = - \Big\{ (n-4)\mathscr{L} + (D_\mu\phi)^* (D^\mu\phi) -2m_o^{\,2} \phi^*\phi - \frac{\lambda_o}{3!} (\phi^*\phi)^2$$

$$+ (n-2)\, \xi_o\, \partial^\rho (A_\rho (\partial . A))\Big\}$$

As is explained earlier, this θ^{μ}_{μ} is not finite even in the one loop order and there is a need for the improvement term in $\theta_{\mu\nu}$. We redefine

$$\theta^{imp}_{\mu\nu} = \theta_{\mu\nu} + \left[\frac{(n-2)}{4(1-n)} + \frac{\tilde{g}}{(1-n)}\right] (\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial^2)\ (\phi^*\phi)$$

and the corresponding improved trace is

$$\theta^{\mu\, imp}_{\mu} = (n-4) \left[- \frac{\lambda_o}{4!} (\phi^*\phi)^2 + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\xi_o}{2} (\partial.A)^2 \right]$$

$$-\frac{(n-2)}{2} \left[\phi^* \frac{\delta S}{\delta\phi^*} + \frac{\delta S}{\delta\phi} \phi\right] -(n-2)\xi_o\ \partial^\rho(A_\rho\ \partial.A)$$

$$+ \tilde{g}\ \partial^2(\phi^*\phi) + m_o^{\ 2}\ \phi^*\phi \qquad (2.7)$$

Green's functions with the insertion of the operators $\phi^* \frac{\delta S}{\delta\phi^*} + \frac{\delta S}{\delta\phi} \phi$ and $\xi_o\ \partial^\rho(A_\rho\ \partial.A)$ are finite [14]. Hence equation (2.7) reduces to

$$\langle\theta^{\mu\, imp}_{\mu}\rangle = \text{finite} + (n-4) \left\langle - \frac{\lambda_o}{4!} (\phi^*\phi)^2 + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\xi_o}{2} (\partial.A)^2\right\rangle$$

$$+ \tilde{g}\ Z_m^{-1} \langle\ \partial^2\ (\phi^*\phi)\ \rangle^R$$

To derive the renormalization properties of the operator $\left(- \frac{\lambda_o}{4!} (\phi^*\phi)^2 + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\xi_o}{2} (\partial.A)^2\right)$, we consider the following set of unrenormalized operators $\left\{0_i\ :\ i=1,2......7\right\}$

$$0_1 = - \frac{\lambda_o}{4!} (\phi^*\phi)^2 + \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\xi_o}{2} (\partial.A)^2$$

$$0_2 = m_o^{\ 2}\ \phi^*\phi$$

$$0_3 = \phi^* \frac{\delta S}{\delta \phi^*} + \frac{\delta S}{\delta \phi} \phi$$

$$0_4 = \frac{\delta S}{\delta A_\mu(x)} A_\mu(x)$$

$$0_5 = \frac{1}{4} F_{\mu\nu}F^{\mu\nu} + \frac{\xi_o}{2} (\partial . A)^2$$

$$0_6 = \frac{\xi_o}{2} (\partial . A)^2$$

$$0_7 = \partial^2 (\phi^*\phi)$$

This set is closed under renormalization. The corresponding renormalization matrix is given by

$$Z_{ij} = \begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ Z_{51} & Z_{52} & Z_{53} & Z_{54} & Z_{55} & Z_{56} & Z_{57} \\ 0 & 0 & Z_{63} & Z_{64} & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & Z_m^{-1} \end{bmatrix}$$

Z_{11} to Z_{16} are simple poles [14] and hence we get

$$\langle \theta_\mu^{\text{imp}\,\mu} \rangle = \text{finite} - \epsilon\, x \,\langle\, \partial^2 \,(\phi^*\phi) \,\rangle^R \tag{2.8}$$

with

$$x = Z_{17} - \frac{1}{\epsilon} \tilde{g}\, Z_m^{-1} \equiv Z_{17} + g\, Z_m^{-1} \tag{2.9}$$

Here Z_{17} is the coefficient of mixing of operator 0_1 with $0_7 = \partial^2(\phi^*\phi)$.

To derive the renormalization group equation for Z_{17}, one uses the expression for the anomalous dimension γ_{ij} of the operator O_i, and the form of the renormalization matrix Z_{ij}. It is given by

$$\left(-\lambda\epsilon + \beta^{\lambda}\right) \frac{\partial Z_{17}}{\partial\lambda} + \left(-\frac{e\epsilon}{2} + \beta^{e}\right) \frac{\partial Z_{17}}{\partial e} - 2\gamma_m Z_{17}$$

$$= Z_{11}\gamma_{17} + Z_{15}\gamma_{57} \qquad (2.10)$$

We are interested in the Lagrange density $\mathcal{L}$ of Eq. (2.4). The results for this Lagrange density are obtained by setting $\lambda = 0$ in the above equation and noting that Z_{17}, being a power series in λ has a smooth limit as $\lambda \to 0$, we arrive at the RG equation satisfied by Z_{17} $(\lambda = 0,\ e^2,\ \epsilon) = Z_{17}$ $(e^2,\ \epsilon)$:

$$\left(-\frac{e\epsilon}{2} + \beta^{e}\right) \frac{\partial Z_{17}}{\partial e} - 2\gamma_m Z_{17} = Z_{11}\gamma_{17} + Z_{15}\gamma_{57} - \beta^{\lambda}\, Y \qquad (2.11)$$

where every quantity in the Eq. (2.11) is evaluated at $\lambda=0$. Here we note that $Y = \left.\frac{\partial Z_{17}}{\partial\lambda}\right|_{\lambda=0}$ is a series in e^2. The quantities Z_{11}, γ_{17}, Z_{15} and γ_{57} which appear on the right hand side of equation (2.11) are given below:

$$Z_{11} = \left(1 - \frac{\beta^{\lambda}}{\lambda\epsilon}\right), \qquad Z_{15} = \frac{\beta^{\lambda}}{\lambda\epsilon} - \frac{2\beta^{e}}{e\epsilon}$$

$$\gamma_{17} = -\frac{\lambda\partial}{\partial\lambda} Z_{17}^{(1)} - \frac{e}{2}\frac{\partial}{\partial e} Z_{17}^{(1)}, \quad \gamma_{57} = -\frac{\lambda\partial}{\partial\lambda} Z_{57}^{(1)} - \frac{e}{2}\frac{\partial}{\partial e} Z_{57}^{(1)}$$

Though Z_{11} and Z_{15} individually contain terms proportional to $1/\lambda$,

however the combination $Z_{11}\gamma_{17} + Z_{15}\gamma_{57}$ does not contain such terms since $Z_{17} - Z_{57}$ is proportional to λ. X of eqn. (2.9) now satisfies an equation (assuming g to depend only on e and ϵ)

$$\left[-\frac{e\epsilon}{2} + \beta^{e}\right] \frac{\partial X}{\partial e} - 2\gamma_m X = Z_{11}\gamma_{17} + Z_{15}\gamma_{57} - \beta^{\lambda} Y$$

$$+ \left(\mu \frac{\partial}{\partial \mu} g \right) Z_m^{-1}$$

$$\equiv Z_{11}\gamma_{17} + Z_{15}\gamma_{57} + T \qquad (2.12)$$

Eq. (2.12) is valid for an arbitrary choice of $g(e^2,\epsilon)$. Now, the first two terms on the right hand side of Eq.(2.12) contain only simple poles, while T may contain double and higher order poles in ϵ. We, now, formulate the necessary and sufficient conditions for the finiteness of $\theta_{\mu}^{\mu\,imp}$.

To this end, we note that finiteness of $\theta_{\mu}^{\mu\,imp}$ isnecessary and sufficient [4] for the finiteness of $\theta_{\mu\nu}^{imp}$. $\theta_{\mu}^{\mu\,imp}$ is finite, as seen from Eq. (2.8), if X has no worse than simple poles in ϵ. We, now, prove the necessary and sufficient conditions for X to have no worse than simple poles.

<u>Theorem</u>: The necessary and sufficient conditions for X to have no worse than simple poles are

(i) T has no worse than simple poles.

(ii) $X^{(2)}$ is made zero.

where X and T have been expanded as

$$X = \sum_{r=-\infty}^{\infty} \frac{X^{(r)}}{\epsilon^{r}} ; \qquad T = \sum_{r=-\infty}^{\infty} \frac{T^{(r)}}{\epsilon^{r}} \qquad (2.13)$$

Proof: (i) First we prove the necessity of the two conditions. Let X have no worse than simple poles. Then $X^{(2)} = 0$. Further, the left hand side of Eq. (2.12) as well as the first two terms on its right hand side have at worst simple poles and hence T can not have worse than simple poles.

(ii) Next, we prove the sufficiency of these conditions.

Suppose that $X^{(2)}$ is zero; and that T has no worse than simple poles. We, then, compare the coefficient of $\epsilon^{-p} (p \geq 2)$ in Eq. (2.12) and obtain

$$- \frac{e}{2} \frac{\partial}{\partial e} X^{(p+1)} + \beta^{e} \frac{\partial X^{(p)}}{\partial e} - 2\gamma_{m} X^{(p)} = 0 \quad p \geq 2 \qquad (2.14)$$

These equations, starting from p=2 successively yield $X^{(3)} = X^{(4)} = \ldots\ldots\ldots 0$, noting that $X^{(p)}(e=0) = 0;\ p \geq 2$. Hence X has no worse than simple poles in ϵ.

Thus, the question is whether g can be chosen so that the two conditions of the theorem above can be fulfilled. In the subsequent sections we try to choose g of either forms of finite improvement program to see whether these two conditions are fulfilled. As a side remark, we wish to point out that in $\lambda\phi^4$ theory, the renormalization group equation satisfied by X, does not have a term which is equivalent to T and thus making $X^{(2)} = 0$ is enough to ensure that all the higher order poles in X are zero.

2.4 Improvement coefficient of the form $\tilde{g}\,(e_0^2\,\mu^{-\epsilon}, \epsilon)$

In this section, we shall consider an improvement coefficient $\tilde{g}$ which is a finite function (at $\epsilon = 0$) of the bare

coupling e_o. Noting that $g = -\frac{\tilde{g}}{\epsilon}$, g has an expansion

$$\tilde{g}\,(e_o^2\,\mu^{-\epsilon},\,\epsilon) = \sum_{n=o}^{\infty} g_n(\epsilon)\,(e_o^2\,\mu^{-\epsilon})^n = \sum_{n=o}^{\infty}\sum_{k=-1}^{\infty} g_n^k\,(e_o^2\,\mu^{-\epsilon})^n\,\epsilon^k \tag{2.15}$$

and consider X order by order.

In $O(e^o)$:

$$X = Z_{17} + \sum_{k=-1}^{\infty} g_o{}^k\,\epsilon^k = Z_{17} + g_o(\epsilon).$$

Now $Z_{17} = 0$ in $O(e^o)$. Hence X has no worse than simple poles.

In $O(e^2)$:

$$X = Z_{17} + g_o(\epsilon)\,Z_m^{-1} + g_1(\epsilon)\,e^2$$

Z_{17}, to $O(e^2)$, vanishes as seen by an explicit calculation. Z_m^{-1} does have a simple pole in $O(e^2)$. The last term in the above has, at worst, a simple pole. Hence the double pole in X is proportional to g_o^{-1}. Thus, X has no worse than simple poles iff $g_o^{-1} = 0$.

In $O(e^4)$:

$$X = Z_{17} + g_o(\epsilon)\,Z_m^{-1} + g_1(\epsilon)\,e^2\,Z_e{}^2\,Z_m^{-1} + g_2(\epsilon)\,e^4$$

Z_{17} has only a simple pole to $O(e^4)$. We expand

$$Z_e^{2q}\;Z_m^{-1} = 1 + \sum_{l=1}^{\infty}\sum_{p=1}^{l}\frac{Y_p^{ql}\;e^{2l}}{\epsilon^p} \tag{2.16}$$

Then, the double poles in X are (noting that $g_o^{-1} = 0$)

$$\frac{e^4}{\epsilon^2}\left[\, g_o^o \;\; Y_2^{o2} + g_1^{-1} \;\; Y_1^{11} \,\right]$$

As seen from appendix A, neither Y_2^{o2} nor Y_1^{11} vanish. Thus, these double poles vanish if

$$g_o^o \;=\; - g_1^{-1} \; \frac{Y_1^{11}}{Y_2^{o2}} \tag{2.17}$$

<u>In $O(e^6)$</u> :

From this order onwards, it is convenient to deal with X indirectly, making use of the theorem in the preceding section. First we require that T has no worse than simple poles:

$$T = \left[\mu \frac{\partial}{\partial\mu} \sum_{n=o}^{\infty} g_n(\epsilon) \; (e_o^2 \, \mu^{-\epsilon} \,)^n\right] Z_m^{-1} - \beta^\lambda \; Y$$

$$= - \epsilon \sum_{n=1}^{\infty} n \; g_n(\epsilon) \; (e_o^2 \, \mu^{-\epsilon} \,)^n \; Z_m^{-1} - \beta^\lambda \; Y$$

$$= - \epsilon \sum_{n=1}^{\infty} n \; g_n(\epsilon) \; e^{2n} \; Z_e^{2n} \; Z_m^{-1} - \beta^\lambda \; Y \tag{2.18}$$

Now β^λ (at $\lambda=0$) is $O(e^4)$. Y to $O(e^2)$ has only a simple pole [14]. Hence the second term on the right hand side has no double poles to this order. To this order the first term is

$$- \epsilon \; g_1(\epsilon) \; e^2 \; (Z_e{}^2 \; Z_m^{-1}) \; - \; 2\epsilon \; g_2(\epsilon) \; e^4 (Z_e{}^4 \; Z_m^{-1}) \; - \; 3\epsilon \; g_3(\epsilon) \; e^6$$

This again has a double pole coming from the first term. It is proportional to g_1^{-1} (with a non zero proportionality constant Y_2^{12}. See Appendix A.)

Hence,

$$g_1^{-1} = 0 \tag{2.19}$$

and Eq. (2.17) then implies that

$$g_o^o = 0 \tag{2.20}$$

Now $X^{(2)}$ to this order is

$$X^{(2)} = Z_{17,3}^{(2)} e^6 + g_o^1 Y_3^{o3} e^6 + g_1^o e^6 Y_2^{12} + g_2^{-1} e^6 Y_1^{21}$$

$\left\{\text{here } Z_{17,q}^{(p)} \text{ is the coefficient of } e^{2q}/\epsilon^p \text{ in } Z_{17}(e^2,\epsilon)\right\}$

$X^{(2)} = 0$ implies

$$g_o^1 Y_3^{o3} = - g_1{}^o Y_2^{12} - g_2^{-1} Y_1^{21} - Z_{17,3}^{(2)} \tag{2.21}$$

<u>In $O(e^8)$</u>:

In this order T is

$$- \epsilon g_1(\epsilon) e^2 (Z_e^2 Z_m^{-1}) -2\epsilon g_2(\epsilon) e^4(Z_e{}^4 Z_m{}^{-1}) -3\epsilon g_3(\epsilon) e^6(Z_e^6 Z_m^{-1})$$

$$- 4\epsilon g_4(\epsilon) e^8 - \beta^\lambda Y$$

As β^λ is $o(e^4)$, $\beta^\lambda Y$ does not have triple poles to this order, nor do the remaining terms [noting Eq. (2.19)]. Equating the double poles in T to zero one obtains

$$- 2g_2^{-1} Y_2^{22} - g_1{}^o Y_3^{13} = a \tilde{Y}_2{}^2 \tag{2.22}$$

where,

$$\beta^\lambda = ae^4 + be^6 + \ldots\ldots$$

and

$$Y = \sum_{r=1}^{\infty} \sum_{p=1}^{r} \frac{\tilde{Y}_p^r \ e^{2r}}{\epsilon^p} \tag{2.23}$$

[It should be noted that $Z_{17}(\lambda, e^2, \epsilon)$ is zero in $O(\lambda)$; hence Y has no terms of $O(e^o)$].

Now, X to this order is

$$X = Z_{17} + g_o(\epsilon)\ Z_m^{-1} + g_1(\epsilon)\ e^2\ Z_e^2\ Z_m^{-1} + g_2(\epsilon)\ e^4 Z_e^4\ Z_m^{-1}$$

$$+ g_3(\epsilon)\ e^6\ Z_e^6\ Z_m^{-1} + g_4(\epsilon)\ e^8$$

Double poles in X cancel if

$$0 = Z_{17,4}^{(2)} + g_o^2\ Y_4^{o4} + g_1^1\ Y_3^{13} + g_2^o\ Y_2^{22}$$

i.e. if

$$g_o^2\ Y_4^{o4} = -\ Z_{17,4}^{(2)} - g_1^1\ Y_3^{13} - g_2^o\ Y_2^{22} \tag{2.24}$$

<u>In $O(e^{10})$</u>:

In this order T is

$$-\epsilon g_1(\epsilon)\ e^2(Z_e^2 Z_m^{-1}) - 2\epsilon\ g_2(\epsilon)\ e^4(Z_e^4 Z_m^{-1}) - 3\epsilon\ g_3(\epsilon)\ e^6(Z_e^6 Z_m^{-1})$$

$$-4\epsilon\ g_4(\epsilon)\ e^8\ (Z_e^8\ Z_m^{-1}) - 5\epsilon\ g_5(\epsilon)\ e^{10} - \beta^\lambda\ Y \tag{2.25}$$

In view of Eqs. (2.19) and (2.20), no term has a quadruple pole. The triple poles in T cancel iff:

$$-\ g_1^o\ Y_4^{14} - 2g_2^{-1}\ Y_3^{23} - a\ \tilde{Y}_3{}^3 = 0 \tag{2.26}$$

The determinant

$$\begin{vmatrix} Y_4^{14} & 2Y_3^{2} \\ Y_3^{13} & 2Y_2^{22} \end{vmatrix} \neq 0$$

(See Appendix A). Hence Eqs. (2.26) and (2.22) uniquely determine g_1^o and g_2^{-1}. Then Eq. (2.21) uniquely determines g_o^1. Double poles in T cancel iff

$$g_1^o\, Y_3^{14} + g_1^1\, Y_4^{14} + 2\, g_2^{-1}\, Y_2^{23} + 2\, g_2^o\, Y_3^{23}$$

$$+ 3\, g_3^{-1}\, Y_2^{32} + a\, \tilde{Y}_2{}^3 + b\, \tilde{Y}_2{}^2 = 0 \qquad (2.27)$$

g_1^o, g_2^{-1} have been determined already and the equation contains three new free parameters g_1^1, g_2^o, g_3^{-1}. We rewrite the equation as

$$g_1^1\, Y_4^{14} + 2\, g_2^o\, Y_3^{23} + 3\, g_3^{-1}\, Y_2^{32} = \text{known} \qquad (2.28)$$

Double poles in X to this order cancel if

$$Z_{17,5}^{(2)} + g_o^2\, Y_4^{o5} + g_o^1\, Y_3^{o5} + g_o^3\, Y_5^{o5} + g_1^o\, Y_2^{14}$$

$$g_1^1\, Y_3^{14} + g_1^2\, Y_4^{14} + g_2^{-1}\, Y_1^{23} + g_2^o\, Y_2^{23} + g_2^1\, Y_3^{23}$$

$$+ g_3^{-1}\, Y_1^{32} + g_3^o\, Y_2^{32} + g_4^{-1}\, Y_1^{41} = 0 \qquad (2.29)$$

In Eq. (2.29), the coefficients g_o^1, g_1^o, g_2^{-1} are already determined. One could eliminate g_1^1 between Eqs. (2.28) and (2.29). One could substitute for g_o^2 in Eqn (2.29) from Eq. (2.24). One then obtains an equation containing the following as yet arbitrary parameters

$$g_o^3 \cdot g_1^2 \cdot g_2^1 \cdot g_3^o \cdot g_4^{-1} \cdot g_2^o \cdot g_3^{-1}$$

Therefore, this constraint can be satisfied in infinite number of ways. Thus the energy momentum tensor <u>can</u> be made finite by the improvement term of the form assumed in this section to $O(e^{10})$.

Next, we shall show that the improvement term of the kind assumed in this section cannot definitely work beyond $O(e^{12})$. To this end we consider the constraints obtained on g_1^o and g_2^{-1} by the requirement that the quartic poles in T in $O(e^{12})$ and quintic poles in T in $O(e^{14})$ must cancel, in particular, if the program is to work in $O(e^{14})$. This places, as we shall see, constraints on the already fixed parameters g_1^o and g_2^{-1} which are inconsistent with those placed by Eqs. (2.22) and (2.26).

That in $O(e^{12})$, the quartic poles in T cancel requires

$$- g_1^o Y_5^{15} - 2 g_2^{-1} Y_4^{24} = a \tilde{Y}_4{}^4 \qquad (2.30)$$

That in $O(e^{14})$, the quintic poles in T cancel requires

$$- g_1^o Y_6^{16} - 2 g_2^{-1} Y_5^{25} = a \tilde{Y}_5{}^5 \qquad (2.31)$$

Consider now the four equations (2.22), (2.26), (2.30) and (2.31). The coefficients occurring in it viz Y_p^{1p} ($p \geq 4$) are related to Y_3^{13} and Y_p^{2p} to Y_2^{22} via renormalization group equations. (See Appendix A). $\tilde{Y}_p{}^p$ are also related ultimately (See Appendix B) to the simple pole divergences in Z_{17} in orders λe^2 and e^4. We have calculated the simple pole terms in $O(\lambda e^2)$ in Z_{17} but not in $O(e^4)$. We shall treat the latter as an unknown. One thus has

four equations in three unknowns, viz:

$$\left[g_o^1 \; Y_3^{13}\right], \; \left[g_2^{-1} \; Y_2^{22}\right] \quad \text{and } e^4/\epsilon \text{ terms in } Z_{17} \,.$$

We have verified that these inhomogeneous equations are inconsistent.

This inconsistency, most probably, is in the set of first three equations [Eqs. (2.22), (2.26), (2.30)] itself. To show this requires a tedious calculation of simple pole divergences in $O(e^4)$ in Z_{17}, which we have not done.

2.5 Improvement coefficient of the form $\tilde{g}\,(e^2,\epsilon)$

In this section, we shall consider an improvement coefficient $\tilde{g}\,(e^2,\epsilon)$ which is a finite function of e^2 at $\epsilon = 0$. Noting that $g = -\frac{\tilde{g}}{\epsilon}$, g has an expansion

$$g\,(e^2,\epsilon) \equiv \sum_{n=o}^{\infty} h_n(\epsilon)\, e^{2n} \equiv \sum_{n=o}^{\infty} \sum_{k=-1}^{\infty} h_n^k\, e^{2n}\, \epsilon^k \tag{2.32}$$

In this case T of Eq. (2.12) is

$$\begin{aligned} T &= -\beta^\lambda Y + \left[\beta^e(e) - \frac{e\epsilon}{2}\right] \frac{\partial}{\partial e}\, g\, Z_m^{-1} \\ &= -\beta^\lambda Y + \left[\frac{\beta^e(e)}{e} - \frac{\epsilon}{2}\right] \sum_{n=1}^{\infty} 2n\, h_n(\epsilon)\, e^{2n}\, Z_m^{-1}. \end{aligned} \tag{2.33}$$

We consider X and T, order by order, as in Sec. 2.4. We shall write, directly, the constraints placed on h_n^k's. They are

$O(e^o)$: no constraint

$$O(e^2) : \quad h_o^{-1} = 0 \tag{2.34}$$

$$O(e^4) : \quad h_o^o \, Y_2^{o2} + h_1^{-1} \, Y_1^{o1} = 0 \tag{2.35}$$

$$O(e^6) : \quad h_1^{-1} \, [\, \beta_3^e \, Y_1^{o1} - \tfrac{1}{2} \, Y_2^{o2} \,] = 0 \tag{2.36}$$

Here β_3^e is the coefficient of $O(e^3)$ term in β^e. As the square bracket is not zero [See Appendix A] Eq. (2.36) and hence Eq. (2.35) imply that

$$h_1^{-1} = 0 = h_o^o \tag{2.37}$$

$O(e^8)$:

$$h_1^o \, [2\beta_3^e \, Y_2^{o2} - Y_3^{o3}] + h_2^{-1} \, [4\beta_3^e \, Y_1^{o1} - 2 \, Y_2^{o2}] - a \, \tilde{Y}_2{}^2 = 0$$

$$h_o^2 \, Y_4^{o4} = - Z_{17,4}^{(2)} - h_1^1 \, Y_3^{o3} - h_2^o \, Y_2^{22} \tag{2.38}$$

$O(e^{10})$:

$$h_1^o \, [2\beta_3^e \, Y_3^{o3} - Y_4^{o4}] + h_2^{-1} \, [4\beta_3^e \, Y_2^{o2} - 2 \, Y_3^{o3}] - a \, \tilde{Y}_3{}^3 = 0 \tag{2.39}$$

Eqs. (2.21) and (2.23) have a unique solution for h_1^o and h_2^{-1}.

$$h_1^1 \, [2\beta_3^e \, Y_3^{o3} - Y_4^{o4}] + 2 \, h_2^o \, [2\beta_3^e \, Y_2^{o2} - Y_3^{o3}] + 3h_3^{-1} [2\beta_3^e \, Y_1^{o1} - Y_2^{o2}]$$

$$= \text{known} \tag{2.40}$$

Eq. (2.40) is analogous to Eq. (2.23) and has infinite number of solutions for h_1^1, h_2^o, h_3^{-1}. In a similar manner, one obtains an equation analogous to Eq. (2.29), which can be satisfied by infinite choices for

$$h_o^3 \, , \; h_1^2 \, , \; h_2^1 \, , \; h_3^o \, , \; h_4^1 \, , \; h_2^o \, , \; h_3^{-1}$$

Thus the energy momentum tensor can be made finite by this kind of improvement program also upto $O(e^{10})$. Next, we shall show in a manner analogous to the previous section that this kind of an improvement program definitely fails beyond $O(e^{12})$.

Cancellation of quartic poles in T to $O(e^{12})$ requires

$$h_1^o \left[2\beta_3^e Y_4^{o4} - Y_5^{o5}\right] + 2 h_2^{-1} \left[2\beta_3^e Y_3^{o3} - Y_4^{o4}\right] = a \tilde{Y}_4{}^4 \tag{2.41}$$

and cancellation of quintic poles in T to $O(e^{14})$ requires

$$h_1^o \left[2\beta_3^e Y_5^{o5} - Y_6^{o6}\right] + 2 h_2^{-1}\left[2\beta_3^e Y_4^{o4} - Y_5^{o5}\right] = a \tilde{Y}_5{}^5 \tag{2.42}$$

As in the previous section, it can be verified that Eqns. (2.38), (2.39), (2.41) and (2.42) are inconsistent equations, proving the failure of this kind of a finite improvement program in $O(e^{14})$ and beyond. As noted in the previous section the breakdown takes place most probably in $O(e^{12})$ itself.

Conclusions:

We had expected that in the special case of scalar QED we dealt with, the FIP would go through since there was scalar field and only one coupling constant like $\lambda\phi^4$ theory. But the results we have found contradict this expectation. We have found that FIP of both kinds I and II do go through to a high order in e^2 i.e. upto $O(e^{10})$. But both of them necessarily fail in $O(e^{14})$ (and most probably in $O(e^{12})$ itself). Thus in both the types of FIP, H_o is independently renormalized atleast from $O(e^{14})$; and

flat space parameters are not sufficient to specify the theory completely in the presence of external gravity from this order. In other words, the only way to obtain finite $\theta_{\mu\nu}$ in this theory that is finite to all orders is to improve $\theta_{\mu\nu}$ by adding an infinite counterterm to it.

CHAPTER - 3

ENERGY MOMENTUM TENSOR IN SCALAR QED AND THE RENORMALIZATION GROUP

3.1 Introduction

It has been established that it is impossible to renormalize the energy momentum tensor (which couples to the gravity) using finite improvement program (FIP) of either kinds, in theories with scalar fields and two coupling constants [14,15]. In the last chapter we found that the finite improvement program does not work in scalar QED even at $\lambda=0$. Hence, the only way to renormalize energy momentum tensor in such theories is by adding an improvement term with divergent coefficient i.e. infinite renormalization of the energy momentum tensor.

It has been discussed in detail in the previous chapter that if the finite improvement program works in the theory, it implies that flat space parameters are sufficient to fix the interaction with the gravity. On the other hand, in the improved energy momentum tensor written below:

$$\theta^{\mathrm{imp}}_{\mu\nu} = \theta_{\mu\nu} - H_o\,(\partial_\mu\partial_\nu - \eta_{\mu\nu}\,\partial^2)\phi^*\phi$$

if the improvement coefficient H_o (a function of the parameters of the theory) contains negative powers of ϵ, as it happens in theories with scalars and two coupling constants, it implies an infinite renormalization of $\theta_{\mu\nu}$. This signals a new parameter (apart from the flat space parameters) in the action corresponding

to the arbitrariness associated with the infinite renormalization, thus needing a new experimental input (related to the `root mean square mass radius' of the scalar particle) [5]. It also means that in such theories, there are infinite number of finite energy momentum tensors which would not be distinguished from each other by a theoretical reason but the correct one is selected only by an experimental input.

The above would be true were it not for the fact that there is another independent theoretical criterion that $\theta^{imp}_{\mu\nu}$ should satisfy, suggested by Collins [8]. This criterion is based on the renormalization group properties of the matrix elements of $\theta^{imp}_{\mu\nu}$. Stated simply, it requires that the Green's functions of $\theta^{imp}_{\mu\nu}$ satisfy a homogeneous renormalization group equation (RGE) with zero anomalous dimension for the operator. In other words, an n point Green's function of $\theta^{imp}_{\mu\nu}$ satisfies the same RGE as the ordinary n-point Green's function i.e., $\theta^{imp}_{\mu\nu}$ is "RG covariant". This has the physical consequences that in the ultraviolet (infrared) limit, the Green's functions for the interaction with an external gravity scale as the ordinary Green's functions do. We are justified in imposing "RG covariance" criterion on $\theta^{imp}_{\mu\nu}$ because in the renormalizable theories without scalars, there exist energy momentum tensors [4] which are finite, finite function of bare quantities and independent of μ when expressed in tersm of bare quantities and carry no parameter in addition to those of the flat space action. The renormalization group equation satisfied by the Green's functions with insertion of such

energy momentum tensors, is homogeneous and with operator anomalous dimension equal to zero. This will be elaborated in section 3.3.

Collins studied "RG covariance" criterion in the context of $\lambda\phi^4$ theory and showed that it is possible to choose a unique energy momentum tensor from a set of an infinite number of them using this criterion. Here, our aim is to study the same for the scalar QED where it becomes especially important and relevant in view of the lack of finite improvement program. If we can fix the unique energy momentum tensor in the theory using "RG covariance" criterion, it implies that no experimental input would be needed though the improvement coefficient does have divergent coefficient. This is the advantage of "RG covariance" criterion.

Here we present the "RG covariance" criterion in the context of scalar QED but we would generalize it to Weinberg-salam model in the next chapter, which we know, is a physically relevant model. Scalar QED is a simpler model to deal with, hence we are taking it up first.

3.2 Preliminaries

We shall work with a complex scalar field coupled to an abelian gauge field described by the Lagrange density

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} (D_\mu \phi)^* (D^\mu \phi) - \frac{1}{2} m_o^2 \phi^* \phi$$

$$- \frac{\lambda_o}{4!} (\phi^* \phi)^2 - \frac{1}{2} \xi_o (\partial . A)^2 \qquad (3.1)$$

where $D_\mu\phi$ is the covariant derivative defined by

$$D_\mu\phi = \partial_\mu\phi - i\, e_o\, A_\mu\phi$$

We shall work with dimensionally regularized quantities and shall use the minimal subtraction (MS) scheme [17].

$\mathcal{L}$ of Eq. (3.1) can be generalized to include minimal interaction with an external gravity $g_{\mu\nu}(x)$ leading to $S_{min}[\phi, A_\mu, g_{\mu\nu}]$. The energy momentum tensor that couples to the gravity, obtained using S_{min} is given by

$$\begin{aligned}\theta_{\mu\nu} &= -\eta_{\mu\nu}\mathcal{L} - F_{\mu\alpha} F^{\alpha}_{\nu} + \frac{1}{2}\left[(D_\mu\phi)^* (D_\nu\phi) + (D_\nu\phi)^* D_\mu\phi\right] \\ &\quad + \xi_o \left[\partial_\mu(\partial.A)\, A_\nu + \partial_\nu(\partial.A)\right]A_\mu - \eta_{\mu\nu}\, \xi_o\, \partial^\rho(\partial.A)\, A_\rho \\ &\quad - \eta_{\mu\nu}\, \xi_o\, (\partial.A)^2 \end{aligned} \qquad (3.2)$$

Since the above $\theta_{\mu\nu}$ is not finite even in one loop order, the improvement term must be introduced to obtain finite $\theta_{\mu\nu}$. As shown in ref. [4,5], divergences in $\theta_{\mu\nu}$ are quadratic in q (the momentum entering via $\theta_{\mu\nu}$).

Here we consider the improved energy momentum tensor containing the improvement term with the most general coefficient H_o. It is given by

$$\theta^{imp}_{\mu\nu} = \theta_{\mu\nu} - H_o\, (\partial_\mu\partial_\nu - \partial^2\, \eta_{\mu\nu})\phi^*\, \phi \qquad (3.3)$$

$$\text{with} \quad H_o \equiv H_o\, (\lambda,\, e^2,\, m^2,\, \xi,\, \epsilon) \qquad (3.4)$$

H_o can contain $\frac{1}{\epsilon}$ term also. To obtain the explicit form for H_o, we consider the expression given below:

$$\left\{\langle\theta_{\mu\nu}\rangle\right\}^{div} = G'(\lambda,\ e^2,\ \xi,\ \epsilon)\ (\partial_\mu\partial_\nu - \partial^2\ \eta_{\mu\nu})\ \langle\phi^*\phi\rangle^R$$

$$= G'(\lambda,\ e^2,\ \xi,\ \epsilon)\ Z_m\ (\partial_\mu\partial_\nu - \partial^2\ \eta_{\mu\nu})\ \langle\phi^*\phi\rangle$$

$$\equiv G\ (\lambda,\ e^2,\ \xi,\ \epsilon)\ (\partial_\mu\partial_\nu - \partial^2\ \eta_{\mu\nu})\ \langle\phi^*\phi\rangle \qquad (3.5)$$

where we have used that $\langle\phi^*\phi\rangle^R = Z_m\ \langle\phi^*\phi\rangle$ where Z_m is defined via $m_o^2 \equiv m^2\ Z_m$ and m is the renormalized mass.

Here by construction G' and hence G $(\lambda,\ e^2,\ \xi,\ \epsilon)$ has only poles in ϵ.

Thus the energy momentum tensor

$$\theta'_{\mu\nu} = \theta_{\mu\nu} - G\ (\lambda,\ e^2,\ \xi,\ \epsilon)\ (\partial_\mu\partial_\nu - \partial^2\ \eta_{\mu\nu})\ \phi^*\phi \qquad (3.6)$$

is a finite energy momentum tensor. But this is not the only one, for one could always add a term of the form

$$- k'\ (\lambda,\ e^2,\ \frac{m^2}{\mu^2},\ \xi,\ \epsilon)\ (\partial_\mu\partial_\nu - \partial^2\ \eta_{\mu\nu})\ \langle\phi^*\phi\rangle^R \qquad (3.7)$$

$$H_o = G\ (\lambda,\ e^2,\ \xi,\ \epsilon) + k'(\lambda,\ e^2,\ \frac{m^2}{\mu^2},\ \xi,\ \epsilon)\ Z_m \qquad (3.8)$$

Now,

$$k'\ (\lambda,\ e^2,\ \frac{m^2}{\mu^2},\ \xi,\ \epsilon) = \bar{k}\ (\lambda,\ e^2,\frac{m^2}{\mu^2},\ \xi) + O(\epsilon) \qquad (3.9)$$

and the last term when substituted in (3.7), its contribution to the e.m. tensor vanishes at n = 4.

Thus, if one is only concerned with $\langle\theta_{\mu\nu}\rangle$ at n=4; we can, without loss of generality, assume

$$H_o = G(\lambda,\ e^2,\ \xi,\ \epsilon) + \bar{k}\ (\lambda,\ e^2,\ \frac{m^2}{\mu^2},\ \xi)\ Z_m \tag{3.10}$$

As already discussed in detail, to remove the arbitrariness in the renormalization of $\theta_{\mu\nu}$, an extra renormalization condition involving a new parameter (not present in the flat space theory) is needed. We introduce this parameter in the equation given below:

$$\bar{k}\ (\lambda,\ e^2,\ \frac{m^2}{\mu^2},\ \xi) \equiv k\ (\lambda,\ e^2,\ \frac{m^2}{\mu^2},\ \xi) + h \tag{3.11}$$

where h is the renormalized coupling constant and is independent of the parameters of the theory (except μ). Thus, we have

$$H_o = G\ (\lambda,\ e^2,\ \xi,\ \epsilon) + \left[k(\lambda,\ e^2,\ \frac{m^2}{\mu^2},\ \xi) + h\right] Z_m \tag{3.12}$$

We know that in the improved energy momentum tensor given by the following equation :

$$\theta^{imp}_{\mu\nu} = \theta_{\mu\nu} + \left[\frac{n-2}{4(1-n)} + \frac{\tilde{g}}{1-n} \right] (\partial_\mu \partial_\nu - \partial^2\ \eta_{\mu\nu})\ \phi^* \phi \tag{3.13}$$

the improvement term $\tilde{g}$ is not necessary [14] upto $O(\lambda^3)$, $O(\lambda e^2)$ and $O(e^4)$. Thus upto these orders,

$$\theta_{\mu\nu} + \frac{n-2}{4(1-n)}\ Z_m^{-1}\ (\partial_\mu \partial_\nu - \eta_{\mu\nu}\partial^2)\ \langle\phi^*\ \phi\rangle^R \tag{3.14}$$

is a finite operator. Comparison of Eqs. (3.3), (3.12) and (3.14) indicates that

$$\text{P.P.}\ \left[\ G\ Z_m^{-1} + \frac{n-2}{4(1-n)}\ Z_m^{-1}\ \right] = 0 \tag{3.15}$$

upto $O(\lambda^3, \lambda e^2, e^4)$. Here P.P. stands for the pole part. Multiplying by $4(1-n)$ and using the finiteness of $\bar{k}$, one obtains

$$\text{P.P.} \left[(G + \bar{k} Z_m) Z_m^{-1} 4(1-n) + (n-2) Z_m^{-1}\right]$$

$$\equiv \text{P.P.} \left[4X Z_m^{-1}(\epsilon - 3) + (2-\epsilon) Z_m^{-1}\right] \qquad (3.16)$$

This relation will be useful in Sec. 3.4.

We define renormalization group quantities:

$$\mu \frac{\partial \lambda}{\partial \mu} \equiv \beta^{\lambda}(\lambda, e^2, \epsilon) = \bar{\beta}(\lambda, e^2) - \lambda\epsilon$$

$$= \lambda\epsilon + \tilde{\beta}_1^{\lambda} \lambda^2 + \beta_2^{\lambda} \lambda e^2 + \beta_1^{\lambda} e^4 + \ldots..$$

$$\mu \frac{\partial}{\partial \mu} e = \beta^e(\lambda, e^2, \epsilon) = \bar{\beta}^e(\lambda, e^2) - \frac{1}{2} e\epsilon \equiv - \frac{1}{2} e\epsilon + \beta_3^e e^3 + \ldots$$

$$- \frac{1}{2} \frac{\mu\partial}{\partial\mu} \mathrm{Ln}\, Z_m \equiv \gamma_m (\lambda, e^2) \equiv \gamma_{m1}\lambda + \gamma_{m2} e^2 + \ldots.$$

$$\mu \frac{\partial}{\partial \mu} \xi = \gamma_{\xi} (\lambda, e^2)\xi \, , \; Z_3^{-1}\mu \frac{\partial}{\partial\mu} Z_3 = \gamma_3(\lambda, e^2)$$

$$Z^{-1}\mu \frac{\partial}{\partial\mu} Z = \gamma (\lambda, e^2, \xi) \qquad (3.17)$$

where Z and Z_3 are the wavefunction renormalizations of the scalar and photon fields.

We quote the needed values

$$\beta_1^{\lambda} = \frac{9}{8\pi^2} \; ; \; \tilde{\beta}_1^{\lambda} = \frac{1}{8\pi^2} \; ; \; \beta_2^{\lambda} = - \frac{1}{\pi^2}$$

$$\gamma_{m1} = \frac{1}{48\pi^2} \equiv - \frac{1}{2} \alpha \; ; \; \gamma_{m2} = - \frac{3}{16\pi^2} \equiv - \frac{1}{2} K$$

$$\beta_3^e = \frac{1}{48\pi^2} \qquad (3.18)$$

Finally we note that, we should, on physical grounds, require that the physical matrix elements of $\theta_{\mu\nu}^{imp}$ should be independent of the gauge parameter ξ. As shown in the Appendix C, this requires that k is ξ -independent. Furthermore as shown in Appendix C, $G(\lambda, e^2, \xi, \epsilon)$ is also independent of ξ.

3.3 RG equation satisfied by $<\theta_{\mu\nu}^{imp}>$ and RG condition on $\bar{k}$

In this section, we shall derive the RG equation satisfied by the proper vertices of $\theta_{\mu\nu}^{imp}$. This RG equation will generally be an inhomogeneous equation, because the improvement coefficient H_o now contains μ dependence, but it can be cast in an apparently homogeneous equation in which an additional term of the kind $\delta \frac{\partial}{\partial h}$ appears in the differential operator acting upon a proper vertex of $\theta_{\mu\nu}^{imp}$ viz. $\Gamma_{\mu\nu}$. In order that $\Gamma_{\mu\nu}$ satisfies an ordinary RG-covariant equation, this term must disappear. This leads to a constraint in the form of a differential equation to be satisfied by $k(\lambda, e^2, \frac{m^2}{\mu^2})$. We have introduced the same mass scales for λ and e^2 for the reasons which will be discussed later. The question of solutions to such an equation satisfied by k, the "boundary conditions" that are appropriate for the existence and uniqueness of its solution are taken up in the subsequent sections.

The derivation of the RG equation for $\Gamma_{\nu\sigma}$ starts as usual, except, now $\theta_{\nu\sigma}^{imp}$ depends explicitly on μ when the improvement coefficient H_o is expressed in terms of bare parameters (We shall allow h to vary with μ though as yet $\mu\frac{\partial h}{\partial \mu}$ is

unspecified, but the RG equation for $\Gamma_{\nu\sigma}$ turns out to be independent of such a term). We express the unrenormalized proper vertices of $\theta_{\nu\sigma}$ as

$$\Gamma_{\nu\sigma} = \Gamma^{(1)}_{\nu\sigma} - H_o(\partial_\nu\partial_\sigma - \eta_{\nu\sigma}\partial^2)\,\Gamma^{(2)} \tag{3.19}$$

where, referring to Eq. (3.3), $\Gamma^{(1)}_{\nu\sigma}$ and $\Gamma^{(2)}$ are the unrenormalized proper vertices of $\theta_{\nu\sigma}$ and $\phi^*\phi$ respectively. As $\theta_{\nu\sigma}$ and $\phi^*\phi$ are functions independent of μ when bare quantities are held fixed,

$$\mu\frac{\partial}{\partial\mu}\;\Gamma^{(1)}_{\nu\sigma}\Big|_{bare} = 0$$
$$\mu\frac{\partial}{\partial\mu}\;\Gamma^{(2)}\Big|_{bare} = 0 \tag{3.20}$$

This leads us to

$$\begin{aligned}\mu\frac{\partial}{\partial\mu}\,\Gamma^{(1)}_{\nu\sigma}\Big|_{bare} &= -\;\mu\frac{\partial}{\partial\mu}\;H_o\Big|_{bare}(\partial_\nu\partial_\sigma - \eta_{\nu\sigma}\partial^2)\,\Gamma^{(2)}\\ &= -\,Z_m^{-1}\,\mu\frac{\partial}{\partial\mu}\;H_o\Big|_{bare}(\partial_\nu\partial_\sigma - \partial^2\eta_{\nu\sigma})\;\times\\ &\quad\times\left[Z_m\,\Gamma^{(2)}\right]\end{aligned} \tag{3.21}$$

Noting that Γ depends on h as

$$\Gamma_{\nu\sigma} = \;.....+\,h\;\;(\partial_\nu\partial_\sigma - \eta_{\nu\sigma}\,\partial^2)\;Z_m\,\Gamma^{(2)}, \tag{3.22}$$

We can write,

$$\mu\frac{\partial}{\partial\mu}\,\Gamma_{\nu\sigma} = \;Z_m^{-1}\;\;\mu\frac{\partial}{\partial\mu}\;\;H_o\Big|_{bare}\,\frac{\partial}{\partial h}\,\Gamma_{\nu\sigma} \tag{3.23}$$

thus turning the inhomogeneous term in Eq. (3.21) into an apparently homogeneous term.

Now consider, for concreteness, a proper vertex with p photon lines and 2q scalar lines. Then

$$\Gamma^{(p,2q)R}_{\nu\sigma} = Z_3^{p/2} \quad Z^q \quad \Gamma^{(p,2q)}_{\nu\sigma} \tag{3.24}$$

The Lorentz indices associated with the photon lines are not indicated. Eq. (3.23) then yields

$$[\mu\frac{\partial}{\partial\mu} + \beta^\lambda(\lambda,\ e^2,\ \epsilon)\ \frac{\partial}{\partial\lambda} + \beta^e(\lambda,\ e^2,\ \epsilon)\ \frac{\partial}{\partial e} + \xi\ \gamma_\xi(\lambda,\ e^2)\ \frac{\partial}{\partial\xi}$$

$$+\ 2\gamma_m\ m^2\ \frac{\partial}{\partial m^2} + \frac{p}{2}\ \gamma_3\ + q\gamma\ + (\mu\ \frac{\partial}{\partial\mu}\ h)\ \frac{\partial}{\partial h}]\ \Gamma^{R(p,2q)}_{\nu\sigma}(\lambda, e^2, m, h, \mu, \xi)$$

$$= Z_m^{-1}\quad \mu\frac{\partial}{\partial\mu}\quad H_o\ \Big|_{bare}\quad \frac{\partial}{\partial h}\ \Gamma^{R(p,2q)}_{\nu\sigma}\ (\lambda, e^2,\ m,\ h,\ \mu,\ \xi)$$

$$= \Big\{Z_m^{-1}\ \ [\beta^\lambda\ (\lambda,\ e^2,\ \epsilon)\ \frac{\partial}{\partial\lambda}\ +\ \beta^e\ (\lambda,\ e^2,\ \epsilon)\ \frac{\partial}{\partial e}]\ G$$

$$-\ 2(h\ +\ k)\ \gamma_m\ \ +\ \ \mu\ \frac{\partial h}{\partial\mu}$$

$$+\ [\ \beta^\lambda\ (\lambda,\ e^2,\ \epsilon)\ \frac{\partial}{\partial\lambda}\ +\ \beta^e\ (\lambda,\ e^2,\ \epsilon)\ \frac{\partial}{\partial e}$$

$$+\ 2(\gamma_m\ -\ 1)\ m^2\ \frac{\partial}{\partial m^2}\ ]k\Big\}\ \frac{\partial}{\partial h}\ \Gamma^{R(p,2q)}_{\nu\sigma}\ (\lambda, e^2,\ m,\ h,\ \mu,\ \xi) \tag{3.25}$$

Eq. (3.25) simplifies to

$$\Big\{\mu\frac{\partial}{\partial\mu} + \beta^\lambda(\lambda,\ e^2,\ \epsilon)\ \frac{\partial}{\partial\lambda}\ +\ \beta^e(\lambda,\ e^2,\ \epsilon)\ \frac{\partial}{\partial e}$$

$$+ \xi \gamma_\xi \frac{\partial}{\partial \xi} + 2\gamma_m m^2 \frac{\partial}{\partial m^2}$$

$$+\left(\frac{p}{2} \gamma_3 + q\gamma \right) + \delta(\lambda, e^2, \frac{m^2}{\mu^2}, \epsilon) \frac{\partial}{\partial h} \Big\} \Gamma^{R(p,2q)}_{\nu\sigma} = 0$$

(3.26)

with

$$\delta(\lambda, e^2, \frac{m^2}{\mu^2}, \epsilon) = Z_m^{-1} \left[\beta^\lambda (\lambda, e^2, \epsilon) \frac{\partial}{\partial \lambda} + \beta^e (\lambda, e^2, \epsilon) \frac{\partial}{\partial e}\right] G$$

$$+ 2(h + k) \gamma_m$$

$$- \left[\beta^\lambda(\lambda, e^2, \epsilon) \frac{\partial}{\partial \lambda} + \beta^e(\lambda, e^2, \epsilon) \frac{\partial}{\partial e} +(2\gamma_m-2)m^2 \frac{\partial}{\partial m^2}\right] k$$

(3.27)

Now all the terms in the Eq. (3.26) except (possibly) the term $\delta \frac{\partial}{\partial h} \Gamma_{\nu\sigma}$ have a finite limit as $\epsilon \to 0$. Hence δ must also be finite at $\epsilon = 0$. Thus putting $\epsilon = 0$, Eq. (3.26) reads

$$\Big\{ \mu \frac{\partial}{\partial \mu} + \bar{\beta}^\lambda \frac{\partial}{\partial \lambda} + \bar{\beta}^e \frac{\partial}{\partial e} + \xi \gamma_\xi \frac{\partial}{\partial \xi} + 2\gamma_m m^2 \frac{\partial}{\partial m^2}$$

$$+\left(\frac{p}{2} \gamma_3 + q\gamma \right) + \bar{\delta} \frac{\partial}{\partial h} \Big\} \Gamma^{R(p,2q)}_{\nu\sigma} = 0$$

with $\bar{\delta} = \delta \left(\lambda, e^2, \frac{m^2}{\mu^2}, \epsilon = 0 \right)$

The above equation implies that the proper vertices of $\theta^{imp}_{\nu\sigma}$ are RG-covariant iff the coefficient of $\frac{\partial}{\partial h} \Gamma_{\nu\sigma}$ vanishes. This requires

$$- \bar{\delta} = 0 = \left[\bar{\beta}^\lambda \frac{\partial}{\partial \lambda} + \bar{\beta}^e \frac{\partial}{\partial e} + (2\gamma_m - 2) m^2 \frac{\partial}{\partial m^2} \right] \bar{k} - 2\bar{k} \gamma_m$$

$$- \left[\lambda \frac{\partial}{\partial\lambda} + \frac{e}{2} \frac{\partial}{\partial e}\right] G^{(1)} (\lambda, e^2) = 0 \qquad (3.28)$$

where $\bar{k}= k+h$; $G^{(1)}$ are the coefficients of the simple pole terms in $G(\lambda, e^2, \epsilon)$ and use has been of the fact that h is a constant independent of λ, e^2, m, ϵ.

Eq. (3.28) is the condition to be satisfied by $\bar{k}$(or equivalently k) and we wish to seek solutions of Eq. (3.28) for k. In general, Eq. (3.28) is a <u>single</u> differential condition on afunction of <u>two</u> independent variables and has an infinite number of solutions. [Collins [8] has suggested that as there are two coupling constants, two mass scales μ_1 and μ_2 be introduced, one for each coupling constant. This yields two differential equations for $\bar{k}$. But this also increases the number of independent variables, because $\bar{k}$ now depends on $x = \mathrm{Ln}\, \frac{\mu_2}{\mu_1}$. Thus one still has one fewer equation than there are independent variables λ, e^2 and x (omitting m). So this does not help]. If we impose a certain kind of "boundary conditions" there will be either no solution, a unique solution or multiple (or infinite) solutions to Eq. (3.28). We wish to discuss physically meaningful "boundary conditions" that yield a unique solution for Eq. (3.28).

We shall briefly comment on the physical significance of the condition (3.28). In renormalizable theories without scalar fields, there exist energy momentum tensors [4] which are (i) finite, (ii) finite functions of bare quantities, (iii) independent of μ when expressed in terms of bare quantities, (iv)

carry no parameters in addition to those of flat space action. For such an energy momentum tensor, the renormalization group equation is homogeneous and with operator anomalous dimension equal to zero. This leads to a certain high energy behavior for the Green's functions of $\theta_{\mu\nu}$ that would enter a gravitational scattering. The condition (3.28) ensures that the energy momentum tensor(s) in scalar QED so obtain leads to a similar RG Equation and a similar high energy behavior for Green's functions.

Alternatively, we could interpret the condition as follows. Unlike in the theories without scalar fields, those with scalar fields, the energy momentum tensor depends on an extra parameter h. The RG condition of Eqn. (3.28) fixes the quantity (k+h) appearing in $\theta^{imp}_{\nu\sigma}$, choosing a particular energy momentum tensor of a set of an infinite number of them.

3.4 Solution Perturbative in λ and e^2

In this section, we shall attempt the most obvious boundary condition on $\bar{k}\,(\lambda,\ e^2,\ \frac{m^2}{\mu^2})$ that (i) it should have a perturbative expansion in powers of λ and e^2,(ii) it should have a finite limit as $m \to 0$. Using the second part of the boundary condition, we shall first simplify Eq. (3.28). It can be rewritten as

$$2(\gamma_m - 1)\,\frac{\partial}{\partial\, \mathrm{Ln}\, m^2}\,\bar{k} \;=\; \left[\lambda\,\frac{\partial}{\partial\lambda} + \frac{1}{2}\,e\,\frac{\partial}{\partial e}\right] G^{(1)}$$

$$+\; 2\bar{k}\,\gamma_m - \left[\bar{\beta}^{\lambda}\,\frac{\partial}{\partial\lambda} + \bar{\beta}^{e}\,\frac{\partial}{\partial e}\right]\bar{k}$$

Consider the above equation at $m = 0$. By the second boundary condition the right hand side has a finite limit at $m = 0$. If it is not zero, then one has

$$2(\gamma_m - 1)\frac{\partial}{\partial \, \mathrm{Ln}\, m^2}\bar{k}\Big|_{m=0} = A(\lambda,\ e^2) \neq 0$$

Thus, $\bar{k} \sim \left[\frac{A(\lambda,\ e^2)}{2(\gamma_m - 1)}\right] \mathrm{Ln}\, m^2 + \ldots$ indicating a singular behavior at $m=0$ contradicting the second part of the boundary condition. Thus one must have $A(\lambda,\ e^2) = 0$; i.e.,

$$(\bar{\beta}^{\lambda}\frac{\partial}{\partial\lambda} + \bar{\beta}^{e}\frac{\partial}{\partial e}) - 2\gamma_m)\ \bar{k}\ (\lambda, e^2) = (\lambda\frac{\partial}{\partial\lambda} + e^2\frac{\partial}{\partial e^2})G^{(1)}(\lambda, e^2)$$

(3.29)

The pole parts in $G(\lambda, e^2, \epsilon)$ are related to those in Z_m^{-1} via the relation (3.16) upto $O(\lambda^3,\ \lambda e^2,\ e^4)$. As this relation contains the combination $X = G + \bar{k}\ Z_m$ it is convenient to transform Eq. (3.29) to express it directly in terms of X rather than G. When this is done, one obtains

$$[(\bar{\beta}^{\lambda} - \lambda\epsilon)\frac{\partial}{\partial\lambda} + (\bar{\beta}^{e} - \frac{1}{2}\ e\epsilon)\frac{\partial}{\partial e}]\ X + \epsilon\lambda\frac{\partial}{\partial\lambda}\bar{k}\ Z_m + \frac{\epsilon e}{2}\frac{\partial}{\partial e}\bar{k}\ Z_m = 0$$

(3.30)

[In obtaining Eq. (3.30), we have made use of the fact, obtainable from the finiteness of δ, that $Z_m^{-1}\ [\beta^{\lambda}\frac{\partial}{\partial\lambda} + \beta^{e}\frac{\partial}{\partial e}]G$ is finite and as G has only poles, it is ϵ independent. Hence it folows that

$$-Z_m^{-1}\ [\beta^{\lambda}\frac{\partial}{\partial\lambda} + \beta^{e}\frac{\partial}{\partial e}]G = (\lambda\frac{\partial}{\partial\lambda} + \frac{e}{2}\frac{\partial}{\partial e})G^{(1)} \qquad (3.31)$$

and this has been used.]

We expand

$$X = X^{(o)} + \tilde{X}^{(1)} \frac{\lambda}{\epsilon} + \hat{X}^{(1)} \frac{\lambda^2}{\epsilon} + \bar{X}^{(2)} \frac{\lambda^2}{\epsilon^2} + \tilde{A}^{(1)} \frac{e^2}{\epsilon}$$
$$+ \hat{A}^{(1)} \frac{e^4}{\epsilon} + \bar{A}^{(2)} \frac{e^4}{\epsilon^2} + \bar{B}^{(1)} \frac{\lambda e^2}{\epsilon} + \bar{B}^{(2)} \frac{\lambda e^2}{\epsilon^2} + \ldots\ldots \tag{3.32}$$

$$Z_m^{-1} = 1 + \frac{\alpha\lambda}{\epsilon} + \frac{\tilde{\alpha}\,\lambda^2}{\epsilon^2} + \frac{d\lambda^2}{\epsilon} + \ldots + k\,\frac{e^2}{\epsilon} + \tilde{k}\;\frac{e^4}{\epsilon^2} + \frac{\tilde{d}\;e^4}{\epsilon} + \ldots\ldots$$
$$+ g\,\frac{\lambda e^2}{\epsilon} + \tilde{g}\,\frac{\lambda e^2}{\epsilon^2} + \ldots\ldots \tag{3.33}$$

$$\bar{k} = \bar{k}_o + \bar{k}_1\lambda + \bar{k}_2 e^2 + \ldots\ldots\ldots \tag{3.34}$$

Here, $\bar{k}$ has been expanded in powers of λ and e^2, as this is the "boundary condition" we are trying. Direct calculation shows that $\bar{k}_o = \frac{1}{6} = X^{(o)}$.

Comparing $O(\epsilon^o)$ terms in Eq. (3.30), we obtain

$$(-\lambda\,\frac{\partial}{\partial\lambda} - \frac{e}{2}\,\frac{\partial}{\partial e})X^{(1)} + \lambda\,\frac{\partial\bar{k}}{\partial\lambda}\,Z_m^{(1)} + \frac{e}{2}\;\frac{\partial\bar{k}}{\partial e}\;Z_m^{(1)} = 0 \tag{3.35}$$

where $X^{(1)}$ and $Z_m^{(1)}$ are the coefficient of simple pole terms in X and Z_m, respectively.

We seek $\bar{k}$ of the form of Eq.(3.34), which satisfies Eq. (3.35). We assume, if possible, that such a $\bar{k}$ exists. Then Eq. (3.35) in $O(\lambda)$ leads to

$$\tilde{X}^{(1)} = 0 \tag{3.36}$$

Further, Eq. (3.30) in $O(\frac{\lambda^2}{\epsilon})$ implies

$$- 2 \bar{X}^{(2)} + \tilde{\beta}_1^{\lambda} \tilde{X}^{(1)} = 0 \qquad (3.37)$$

This, together with Eq. (3.36) implies

$$\bar{X}^{(2)} = 0 \qquad (3.38)$$

Now, consider Eq. (3.16) in $O(\frac{\lambda^2}{\epsilon})$ leads to

$$4\tilde{\alpha}X^{(o)} - 12\hat{X}^{(1)} - 12d\, X^{(o)} + 4\tilde{X}^{(1)}\alpha + 4\bar{X}^{(2)} + 2d-\alpha = 0 \qquad (3.39)$$

Using Eqs. (3.36), (3.38) and the value $X^{(o)} = \frac{1}{6}$ leads one to

$$\hat{X}^{(1)} = - \frac{1}{36} \tilde{\alpha} \qquad (3.40)$$

From RGE satisfied by Z_m^{-1}, one obtains

$$\tilde{\alpha} = \frac{1}{2} \alpha \, (\tilde{\beta}_1^{\lambda} + 2\gamma_{m1}) \qquad (3.41)$$

and thus one has

$$\hat{X}^{(1)} = - \frac{1}{72} \alpha \, (\tilde{\beta}_1^{\lambda} + 2\gamma_{m1}) \qquad (3.42)$$

Eq. (3.35), in $O(e^2)$, gives

$$\tilde{A}^{(1)} = 0 \qquad (3.43)$$

Eq. (3.30) [or alternately Eq. (3.16) also] implies

$$\bar{A}^{(2)} = 0 \qquad (3.44)$$

Eq. (3.16), in $O(\frac{e^4}{\epsilon})$, yields

$$4X^{(o)}\,\tilde{k} - 12\tilde{d}\,X^{(o)} + 4\tilde{A}^{(1)}\,k - 12\hat{A}^{(1)} + 4\bar{A}^{(2)} + 2\tilde{d} - \tilde{k} = 0 \tag{3.45}$$

In view of Eqs. (3.43) and (3.44) and the value of $X^{(o)}$, this simplifies to

$$\hat{A}^{(1)} = -\frac{1}{36}\,\tilde{k} \tag{3.46}$$

RGE for Z_m^{-1} yields,

$$\tilde{k} = \frac{1}{2}\left[\beta_1^{\lambda}\,\alpha + k(2\beta_3^{e} + 2\gamma_{m2})\right] \tag{3.47}$$

so that

$$\hat{A}^{(1)} = -\frac{1}{72}\left[\beta_1^{\lambda}\,\alpha + k(2\beta_3^{e} + 2\gamma_{m2})\right] \tag{3.48}$$

In a similar manner, Eq. (3.30) yields

$$\bar{B}^{(1)} = -\frac{1}{72}\,(\beta_2^{\lambda}\,\alpha + 2\gamma_{m1}\,k + 2\gamma_{m2}\,\alpha) \tag{3.49}$$

Now, we obtain the relations that must be satisfied by $\bar{k}_1$ and $\bar{k}_2$. The crucial point is that there are only <u>two</u> variables and they must satisfy <u>three</u> equations obtained from Eq. (3.35) in orders λ^2, λe^2 and e^4. This requires that they be consistent. The consistency requires a condition to be satisfied by calculable coefficients of the Eq. (3.53).

Eq. (3.35) in $O(\lambda^2)$, yields

$$\bar{k}_1 = -\frac{2\,\hat{X}^{(1)}}{\alpha} = \frac{1}{36}\,(\tilde{\beta}_1^{\lambda} + 2\gamma_{m1}) \tag{3.50}$$

Eq. (3.35), in $O(e^4)$, yields

$$\bar{k}_2 = - \frac{2\,\hat{A}^{(1)}}{k} = \frac{1}{36} \left[\beta_1^\lambda \frac{\alpha}{k} + 2\beta_3^e + 2\gamma_{m2} \right] \tag{3.51}$$

Eq. (3.35), in $O(\lambda e^2)$, yields

$$\bar{k}_2 \alpha + \bar{k}_1 k = - 2\, \bar{B}^{(1)} = \frac{1}{36} (\beta_2^\lambda \alpha + 2\gamma_{m1} k + 2\gamma_{m2} \alpha) \tag{3.52}$$

As is easily verified, these equations are consistent iff,

$$\beta_2^\lambda \alpha - \tilde{\beta}_1^\lambda k - \beta_1^\lambda \frac{\alpha^2}{k} - 2\, \beta_3^e \alpha = 0 \tag{3.53}$$

With the values given in Sec.3.2, it is easily verified that this condition is not met. Hence, no solution for $\bar{k}$ at $m = 0$. Thus, the obvious perturbative boundary condition yields no solution for $\bar{k}$.

3.5 Solution perturbative around a nontrivial fixed point $e^*, \lambda^* \neq 0$

As Collins [8] has commented, it is more natural to impose boundary conditions at a fixed point. We assume the existence of a nontrivial fixed point $\lambda^*, e^* \neq 0$; and require that the solution for $\bar{k}\,(\lambda, e^2, \frac{m^2}{\mu^2})$ be perturbative in powers of $(\lambda - \lambda^*)$ and $(e^2 - e^{*2})$ and that it is analytic in m^2.

We first consider the part of $\bar{k}$ independent of m. To show that a unique solution for $\bar{k}\,(\lambda, e^2) \equiv \bar{k}\,(\lambda, e^2, m = 0)$ is obtained, we expand

$$\bar{k}\,(\lambda, e^2) = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} k_{mn} (\lambda - \lambda^*)^m (e^2 - e^{*2})^n$$

$$\overline{\beta}^{\lambda}(\lambda,\ e^2) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \overline{\beta}^{\lambda}_{mn}\ (\lambda-\lambda^*)^m\ (e^2 - e^{*2})^n$$

$$2e\ \overline{\beta}^{e}(\lambda,\ e^2) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \overline{\beta}^{e}_{mn}\ (\lambda-\lambda^*)^m\ (e^2 - e^{*2})^n$$

$$\gamma^m(\lambda,\ e^2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \gamma^m_{pq}\ (\lambda-\lambda^*)^p\ (e^2 - e^{*2})^q$$

$$(\lambda \frac{\partial}{\partial\lambda} + e^2 \frac{\partial}{\partial e^2})\ G^{(1)}(\lambda,\ e^2) \equiv \xi(\lambda, e^2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \xi_{mn}\ (\lambda-\lambda^*)^m\ (e^2-e^{*2})^n$$

$$(3.54)$$

We wish to consider the equation

$$(\overline{\beta}^{\lambda} \frac{\partial}{\partial\lambda} + 2e\ \overline{\beta}^{e} \frac{\partial}{\partial e^2} - 2\gamma_m)\ \overline{k}\ (\lambda,\ e^2) = \xi\ (\lambda,\ e^2)$$

$$(3.55)$$

The proof that a unique solution for $\overline{k}$ of theform of the first of Eq. (3.54) exists is very straightforward.

Consider Eq. (3.55) in $O\ (\lambda - \lambda^*)^o\ (e^2 - e^{*2})^o$. One has

$$- 2\ \gamma^m_{oo}\ \ \overline{k}_{oo} = \xi_{oo} \tag{3.56}$$

Assuming that $\gamma^m_{oo} = \gamma^m(\lambda^*) \neq 0$, one has

$$\overline{k}_{oo} = - \frac{1}{2}\ \frac{\xi_{oo}}{\gamma^m_{oo}} \tag{3.57}$$

Now we proceed by induction. Let us assume that k_{pq} for $p+q \leq N$, have been uniquely fixed via Eq. (3.55). We wish to show that k_{pq} with $p + q = N+1$ can be fixed via Eq. (3.55). To this end, we consider Eq.(3.55) in $O(\lambda-\lambda^*)^p\ (e^2-e^{*2})^q$ with $0 \leq p,q \leq N + 1$, p+q

$= N + 1$. One has

$$(p\vec{\beta}^{\lambda}_{10} + q\,\bar{\beta}^{e}_{01} - 2\gamma^{m}_{00})\, k_{pq} + \text{terms known or already fixed uniquely}$$

$$= \xi_{pq} \tag{3.58}$$

This fixed k_{pq} uniquely (unless $p\,\vec{\beta}^{\lambda}_{10} + q\,\bar{\beta}^{e}_{01} - 2\,\gamma^{m}(\lambda^*) = 0$ accidentally). These conditions cannot be verified as nothing is known about λ^* and $\vec{\beta}^{\lambda}_{10}$ etc. But for such a condition to be satisfied is expected only as an accident. As the above proof is valid for any p, q with $p + q = N + 1$ $0 \le p, q \le N + 1$, k_{pq} for <u>all</u> $p + q = N + 1$ are determined. As the assumption made applies to $N = 0$ via Eq. (3.57), the result is proved by induction; fixing $\bar{k}(\lambda, e^2)$ uniquely. Now let

$$\bar{k}(\lambda, e^2, \frac{m^2}{\mu^2}) = \bar{k}(\lambda, e^2) + \sum_{n=1}^{\infty} (\frac{m^2}{\mu^2})^n\, \bar{k}_n(\lambda, e^2) \tag{3.59}$$

From Eqs. (3.28) and (3.29), it is easily seen that

$$\left\{\vec{\beta}^{\lambda} \frac{\partial}{\partial\lambda} + \bar{\beta}^{e} \frac{\partial}{\partial e} + \left[(2n-2)\gamma_m - 2n\right]\right\} \bar{k}_n = 0 \quad n \ge 1 \tag{3.60}$$

Following a similar procedure used in obtaining the unique solution for $\bar{k}(\lambda, e^2)$, one easily sees that the only perturbative $[\text{in } (\lambda - \lambda^*), (e^2 - e^{*2})]$ solution to Eqs. (3.60) are

$$\bar{k}_n(\lambda, e^2) = 0; \quad n \ge 1 \tag{3.61}$$

Hence our boundary conditions fix $\bar{k}(\lambda, e^2, \frac{m^2}{\mu^2})$ uniquely.

We have explored the possible alternate boundary conditions too that fix a unique solution to the equation (3.28) [18]. These boundary conditions are of two kinds. One, in which $\bar{k}$ be perturbative in λ only and second, in which $\bar{k}$ be perturbative in e^2 only. Unlike the boundary conditions used earlier, the physical significance of these boundary conditions is however not clear.

3.6 Less restrictive 'RG covariance' condition

In section (3.4) it was shown that when we impose perturbative boundary condition on $\bar{k}$ in equation (3.28), no solution is possible. It happened because when equation (3.28) was considered in different orders in coupling constants, the number of independent $\bar{k}_n$'s that appeared in these equations, was less than the number of constraint equations on them. We recollect that equation (3.28) arises when we put $\bar{\delta} = 0$, so as to make equation (3.26) 'RG Covariant'. We shall now consider a condition, somewhat weaker than 'RG covariance', which will allow for a perturbative solution to equation (3.28). The equation to be considered is written below:

$$- \bar{\delta} = \left(\bar{\beta}^{\lambda} \frac{\partial}{\partial \lambda} + \bar{\beta}^{e} \frac{\partial}{\partial e}\right)\bar{k} - 2\gamma_m \bar{k} - \left[\lambda \frac{\partial}{\partial \lambda} + \frac{e}{2} \frac{\partial}{\partial e}\right] G^{(1)}(\lambda, e^2) \qquad (3.62)$$

which leads to

$$\left[\left(\bar{\beta}^{\lambda} - \lambda\epsilon\right) \frac{\partial}{\partial \lambda} + \left(\bar{\beta}^{e} - \frac{e\epsilon}{2}\right) \frac{\partial}{\partial e}\right] \times + \left(\epsilon\lambda \frac{\partial}{\partial \lambda} \bar{k}\right) Z_m + \frac{e\epsilon}{2} \left(\frac{\partial}{\partial e} \bar{k}\right) Z_m$$

$$= - \bar{\delta} Z_m$$

In $O(\epsilon^{0})$ the above equation gives

$$- \lambda \frac{\partial}{\partial\lambda} X^{(1)} - \frac{e}{2}\frac{\partial}{\partial e} X^{(1)} + (\lambda \frac{\partial}{\partial\lambda} \bar{k}) Z_m^{(1)} + (\frac{e}{2} \frac{\partial}{\partial e} \bar{k}) Z_m^{(1)} = - \bar{\delta} \tag{3.63}$$

Equations (3.50), (3.51) and (3.52) get changed, with now $\bar{\delta}$ also present on the right hand side of these equations.

When equation (3.62) is considered in $O(\lambda)$ and $O(e^2)$, $\bar{\delta}$ can be set to zero by fixing $\bar{k}_0$ as $\frac{1}{6}$. Also in equations (3.50) and (3.52) $\bar{k}_1$ and $\bar{k}_2$ can always be so chosen that $\bar{\delta} = 0$.

In general, it is shown in the appendix D that it is possible to choose $\bar{k}_n$'s appearing in the equations in $O(\lambda^{m+1}, \lambda^p (e^2)^q)$ where m,p,q all run from 1 to ∞, such that $\bar{\delta} = 0$ in $O(\lambda^{m+1}, \lambda^p (e^2)^q)$. This way $\bar{k}_n$'s get fixed uniquely and hence $\theta^{imp}_{\nu\sigma}$. The first nonzero contribution to $\bar{\delta}$ comes in $O(e^4)$. We have determined the magnitude of $\bar{\delta}$ in $O(e^4)$ using equations (3.50), (3.52) and (3.63) in $O(e^4)$, by substituting the values of different calculable quantities. The physical significance of '$\bar{\delta}$' being non zero can be understood, provided $\lambda >> e^2$. We compare the contribution from a typical term $\bar{\beta}^{\lambda} \frac{\partial}{\partial\lambda} \Gamma_{\theta_{\nu\sigma}}imp$ with $\bar{\delta} \frac{\partial}{\partial h} \Gamma_{\theta_{\nu\sigma}}imp$ in the RG equation satisfied by $\Gamma_{\theta_{\nu\sigma}}imp$. For the sake of concreteness we consider these terms in $O(\lambda^2)$, $O(\lambda e^2)$ and $O(e^4)$. We also assume that $\frac{\lambda}{4\pi} << 1$ such that the perturbative expansion can be carried out but it is much larger than $\frac{e^2}{4\pi}$. Under such conditions we find that the contribution in $O(e^4)$ coming from $\bar{\delta} \frac{\partial}{\partial h}\Gamma_{\theta_{\nu\sigma}}imp$ term, is much smaller than $O(\lambda^2)$ and even $O(\lambda e^2)$

contribution from $\vec{\beta}^{\lambda} \frac{\partial}{\partial \lambda} \Gamma_{\theta_{\nu\sigma}}\text{imp}$.

We know that the coefficient of $\frac{\partial}{\partial \lambda} \Gamma_{\theta_{\nu\sigma}}\text{imp}$ in the RG equation satisfied by $\Gamma_{\theta_{\nu\sigma}}\text{imp}$ is $\mu \frac{\partial \lambda}{\partial \mu}$, which we define as $\beta^{\lambda}(\lambda, e^2, \epsilon)$. In the same way, we may interpret δ, the coefficient of $\frac{\partial}{\partial h} \Gamma_{\theta_{\nu\sigma}}\text{imp}$ as $\mu \frac{\partial h}{\partial \mu}$. In that case, the results that we have mentioned in the paragraph above, imply that as the mass scale μ is changed, the variation in $\Gamma_{\theta_{\nu\sigma}}\text{imp}$ caused due to the change in λ is much larger than that caused by the change in h. We'll indicate in Chapter 5 in Sec. 5.7 that this situation can persist near the nontrivial UV fixed point of the theory, if certain conditions are met.

The case of obtaining a unique $\theta_{\nu\sigma}^{\text{imp}}$, with $\bar{\delta} \neq 0$, will be taken up in the subsequent chapter to obtain some non trivial results in the context of high energy behavior of $\Gamma_{\theta_{\nu\sigma}}\text{imp}$ in theories with scalars and more than one coupling constant.

Conclusion:

We have shown that 'RG covariance' criterion (together with certain boundary conditions) when applied at a non trivial fixed point λ^*, e^* ($\neq$ 0) of the scalar QED theory, uniquely determines the energy momentum tensor. This unique energy momentum tensor may couple to the external gravity. Also we showed that 'RG covariance' criterion when taken perturbatively in λ and e^2 did not yield any unique energy momentum tensor. But we did obtain a unique energy momentum tensor when 'δ' was made zero, not in all orders in λ and e^2 but only in selected orders as shown in the last section, where we also discussed the physical significance of 'δ' being non zero.

CHAPTER - 4

RENORMALIZATION OF ENERGY MOMENTUM TENSOR IN THEORIES WITH SPONTANEOUS SYMMETRY BREAKING

4.1 Introduction

We shall first recapitulate what we mean by 'spontaneous symmetry breaking' in a quantum field theory. For the sake of simplicity we shall consider $\lambda\phi^4$ theory, given by the lagrange density

$$\mathcal{L} = \frac{1}{2} (\partial_\lambda \phi)^2 - \frac{\mu_o^2}{2} \phi^2 - \frac{\lambda_o}{4!} \phi^4$$

It is easily seen that for $\mu_o^2 > 0$, the effective potential has a minimum at $\phi = 0$. But for $\mu_o^2 < 0$ the ground state expectation value of field is non vanishing and its value is $\pm \sqrt{\frac{-6\mu_o^2}{\lambda_o}}$. In field theory, the ground state is the vacuum and thus we see that if $\mu_o^2 < 0$, the vacuum expectation value (VEV) of the field is not zero and has the value $v = \pm \sqrt{\frac{-6\mu_o^2}{\lambda_o}}$ to zeroth order in perturbation theory. These two vacuum expectation values of the field ϕ are symmetric about $\phi = 0$ and are degenerate. When we choose one of the possible vacuum to build the theory, the original reflection symmetry of the theory ($\phi \rightarrow - \phi$), is broken. The vacuum now does not respect this symmetry. This is known as the <u>spontaneous symmetry breaking</u>. If the lagrangian is symmetric

but the physical vacuum is not invariant under the symmetry group, then such a symmetry is called spontaneously broken symmetry. The lagrange density in the broken theory expressed in terms of the field variable $\phi'(=\phi-v)$, which has the zero VEV is given by

$$\mathcal{L}' = \frac{1}{2} (\partial_\lambda \phi')^2 - (-\mu_o^2 \phi'^2) - \frac{\lambda_o V \phi'^3}{3!} - \frac{\lambda_o}{4!} \phi'^4$$

(Upto certain constant which does not alter Euler Lagrange equation of motion).

The scalar particle now has mass $(-2 \mu_o^2)$. This lagrangian is not symmetric under the transformation $\phi' \to - \phi'$ in an obvious way but the symmetry is hidden in the sense that when ϕ' is expressed in terms of ϕ, we get back to $\mathcal{L}$, which is ofcourse symmetric. Here we have talked of $\lambda\phi^4$ theory having reflection (discrete) symmetry along with the spontaneously broken symmetry. There are other theories in which lagrangian has local symmetry as well as the spontaneously broken symmetry. A good example is Weinberg-Salam model in which we know that spontaneous symmetry breaking has an important role to play [23].

The renormalization of $\lambda\phi^4$ theory with the spontaneous symmetry breaking has been dealt with, by explicitly calculating the divergent diagrams that arise in the broken theory at the one loop level in perturbation theory (in coupling constant) [23]. They give rise to the shift in the VEV of the field ϕ, leading to the redefinition of the already shifted field and this would take place at each loop level. It has been shown that the counterterms

required for the divergences at the one loop level are the same as that for the theory without SSB. In other words, the ultraviolet divergences respect the symmetry of the lagrangian even if the vacuum does not. It is because the ultraviolet divergences are insensitive to finite mass scales and we know that spontaneous symmetry breaking is generated by the nonzero VEV of ϕ which has the dimensions of mass.

In the work presented here, we shall discuss spontaneously broken theories using generating functionals of connected Green's functions and proper vertices. The advantage of this formalism is that the results obtained are valid to all orders in perturbation theory. Moreover we can deal with the perturbation theory in the loop expansion parameter rather than perturbation in the coupling constant which was discussed in the last paragraph. The results of the perturbation theory in terms of loop expansion parameter remain the same, even when ϕ acquires a non zero VEV because loop expansion is independent of the shift in the field ϕ in the lagrangian. Using this formalism, it has been shown, in the context of σ model [22], that the process of renormalization does not induce additional symmetry breaking in the sense that the symmetric counterterms are sufficient to remove the infinities from the theory whether or not the symmetry is broken externally (i.e. the term that breaks the symmetry explicitly present in the lagrangian) or internally as is the case we discussed in the beginning of this chapter.

Here we are interested in the renormalization of energy

momentum tensor in the theories with spontaneous symmetry breaking. We shall consider $\lambda\phi^4$ theory, to begin with. We know that in the unbroken $\lambda\phi^4$ theory, it is possible to renormalize energy momentum tensor using `finte improvement program' [8]. As mentioned earlier, the improved energy momentum tensor of the form

$$\theta_{\mu\nu}^{imp} = \theta_{\mu\nu} + H_0(\epsilon)\ (\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2$$

is finite to all orders in perturbation theory. $H_0(\epsilon)$ is a series in non negative powers of ϵ = 4-n and is chosen uniquely in each order in perturbation theory. We expect that it should be possible to make $\theta_{\mu\nu}$ finite using finite improvement program in broken $\lambda\phi^4$ theory as well since the ordinary Green's functions in the broken $\lambda\phi^4$ theory can be made finite without requiring any extra counterterms, as discussed above. We shall investigate it in the first part of the work.

Next we take up the renormalization of energy momentum tensor in Weinberg-Salam model which is the renormalizable theory of electroweak interactions in which the intermediate vector boson masses are generated by the spontaneous symmetry breaking (Higgs mechanism) and not inserted by hand. Since it is the theory with scalars and more than one coupling constant, we know that the `finite improvement program' would not go through [14,15]. We shall make use of the result obtained in the first part of the work and deal with the unbroken form of W-S model. We shall renormalize the energy momentum tensor in this theory by adding an infinite counterterm to it and to fix the finite energy momentum

tensor uniquely, we would use 'RG covariance' criterion as done for scalar QED. There we discussed in detail that it is a theoretical criterion which enables us to pick up the unique energy momentum tensor from a set of finite energy momentum tensors. We are interested in renormalizing the energy momentum tensor in W-S model because it is more realistic and physically relevant than scalar QED, or $\lambda\phi^4$ theory. We wish to see if the results obtained for the scalar QED in the last chapter are valid for this model as well. These results have importance when we will consider the high energy behavior of the Green's functions with the insertion of energy momentum tensor inscalar QED and W-S model in the next chapter.

4.2 Finite improvement program in spontaneously broken $\lambda\phi^4$ theory

4.2.1 Preliminaries

In this section, we shall take up the renormalization of energy momentum tensor in the $\lambda\phi^4$ theory with spontaneous symmetry breaking, using the formalism of generating functionals. First, let us briefly go through some of the useful definitions.

The generating functional of connected Green's functions Z[J] in $\lambda\phi^4$ theory, is defined by the expression given below:

$$W[J] = \exp\,[iZ[J]] = \int[d\phi]\,\exp\left\{i\int d^4x\,\left\{\mathcal{L}(\phi,\partial_\mu\phi(x)) + J(x)\phi(x)\right\}\right\}$$

The first derivative of Z[J] with respect to J(x) is

$$\frac{\delta Z[J]}{\delta J(x)} = \frac{1}{W[J]} \int [d\phi]\ \phi(x)\ \exp\left\{ i \int d^4x\ (\mathcal{L} + J(x)\ \phi(x)) \right\} = \Phi(x)$$

where $\Phi(x)$ is the expectation value of $\phi(x)$ in the presence of the source $J(x)$ i.e. it is the classical field.

We define

$$\left.\frac{\delta Z[J]}{\delta J(x)}\right|_{J=0} = v$$

In the spontaneously broken theories $v \neq 0$ as we have already discussed. It is independent of x.

The generating functional of the proper vertices $\Gamma[\Phi]$, is Legendre's transform of $Z[J]$ and is given by

$$\Gamma[\Phi] = Z[J] - \int dx\ J(x)\ \Phi(x) \qquad (1)$$

$$\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)} = -\ J(x)$$

$$\text{and} \quad \left.\frac{\delta\Gamma[\Phi]}{\delta\Phi(x)}\right|_{\Phi=v} = 0$$

i.e. when $J(x) = 0$, Φ takes the value v.

The generating functional defined in equation (1) when Taylor expanded about $\phi = v$, has the representation:

$$\Gamma[\Phi] = \Gamma(v) + \frac{1}{2!} \int \left.\frac{\delta^2\Gamma[\Phi]}{\delta\Phi(x_1)\delta\Phi(x_2)}\right|_{\Phi=v} (\Phi(x_1)-v)\ (\Phi(x_2)-v)\,d^4x_1 d^4x_2$$

$$+\frac{1}{3!}\int \left.\frac{\delta^3\Gamma[\Phi]}{\delta\Phi(x_1)\delta\Phi(x_2)\delta\Phi(x_3)}\right|_{\Phi=v} (\Phi(x_1)-v)(\Phi(x_2)-v)(\Phi(x_3)-v)\,d^4x_1 d^4x_2 d^4x_3$$

$$+ \ \cdots\cdots\cdots\cdots\cdots\cdots$$

We define $\Gamma^{(n)}_{12....n}(v) = \left.\frac{\delta^n \Gamma[\Phi]}{\delta\Phi(x_1)\ldots\delta\Phi(x_n)}\right|_{\Phi=v}$

which are the proper vertices of broken $\lambda\phi^4$ theory. Hence

$$\Gamma[\Phi] = \Gamma[v] + \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^{(n)}_{12....n}(v)\, (\Phi_1 - v)\ldots(\Phi_n - v)$$

(Summation integration convention understood)

The above relation tells us that if we expand $\Gamma[\Phi]$ about $\Phi = v$, we get the proper vertices of the broken theory. Also we have a relation of the kind

$$\Gamma^{(n)}_{12...n}(v) = \sum_{m=0}^{\infty} \frac{1}{m!} v^m \, \Gamma^{(n+m)}_{12...n} \; (v=0) \qquad (2)$$

It is the relation between the proper vertices of the broken and the unbroken theory. Once $\Gamma[\Phi]$ is made finite in the unbroken theory, the above relation ensures that $\Gamma^{(n)}_{12...n}(v)$ are finite too. It is because when $\Gamma^R[\Phi^R]$ is expanded about $\Phi^R = 0$, the expansion coefficients are n point proper vertices of the unbroken theory which are finite since $\Gamma^R[\Phi^R]$ is finite. The finiteness of $\Gamma^{(n)}_{12...n}(v^R)$ then follows from the relation (2).

4.2.2 With this much background, we proceed to define the generating functional from which the Green's functions with one insertion of $\theta^{imp}_{\mu\nu}$ can be obtained ($\theta^{\mu\nu\, imp}$ is the finite energy momentum tensor in the unbroken $\lambda\phi^4$ theory where we

have used the finite improvement program. It has been defined in the beginning of this chapter). It is given by

$$W[J,h_{\mu\nu}] = \exp[i\ Z[J,h_{\mu\nu}]]$$

$$= \int D\phi\ e^{iS_0 + \frac{1}{2}\int h_{\mu\nu}\theta^{\mu\nu\, imp} + \int J(x)\ \phi(x)\ dx}$$

where $h^{\mu\nu}$ is the source for $\theta^{imp}_{\mu\nu}$ and S_0 is the action in the unbroken $\lambda\phi^4$ theory.

Here $$\left.\frac{\delta Z[J,h_{\mu\nu}]}{\delta J(x)}\right|_{\substack{J=0\\ h_{\mu\nu}=0}} = \left.\Phi(x)\right|_{\substack{J=0\\ h_{\mu\nu}=0}} = v$$

We define

$$\Gamma\ [\Phi,\ h_{\mu\nu}\ ] = Z\ [J,h_{\mu\nu}\ ] - \int J(x)\ \Phi(x)\ dx$$

Φ and $h_{\mu\nu}$ are taken to be two independent variables. In the unbroken $\lambda\phi^4$ theory, the relations written below hold

$$W\ [\ J,\ h_{\mu\nu}\ ] \equiv W^R\ [\ J^R,\ h_{\mu\nu}\ ]$$

$$Z\ [\ J,\ h_{\mu\nu}\ ] \equiv Z^R\ [\ J^R,\ h_{\mu\nu}\ ]$$

$$\Gamma\ [\ \Phi,\ h_{\mu\nu}\ ] \equiv \Gamma^R\ [\ \Phi^R,\ h_{\mu\nu}\ ]$$

Since $\theta^{imp}_{\mu\nu}$ is a finite operator in the unbroken $\lambda\phi^4$ theory, $h^{\mu\nu}$ does not need renormalization.

In the unbroken theory, we have

$$\Gamma^R \left[\Phi^R, h^{\mu\nu}\right] = \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^{(n)}_{12..n} \left[h^{\mu\nu}\right] \Phi^R_1 \ldots \Phi^R_n \tag{4}$$

$$\text{where } \Gamma^{(n)}_{12..n} \left[h^{\mu\nu}\right] = \left. \frac{\delta^n \Gamma^R \left[\Phi^R, h^{\mu\nu}\right]}{\delta\Phi^R_1 \ldots \delta\Phi^R_n} \right|_{\Phi^R = 0}$$

(Summation integration convention understood).

The quantity $\left. \frac{\delta \Gamma^R \left[\Phi^R, h^{\mu\nu}\right]}{\delta h^{\mu\nu}} \right|_{h^{\mu\nu} = 0}$ is the generating functional for the renormalized proper vertices with the insertion of $\theta^{imp}_{\mu\nu}$, in the unbroken $\lambda\phi^4$ theory.

As in the case of ordinary Green's functions, in the broken $\lambda\phi^4$ theory, we can write down the Taylor's expansion for $\Gamma^R \left[\Phi^R, h_{\mu\nu}\right]$ about $\Phi^R = v^R$. In the broken $\lambda\phi^4$ theory, we have

$$\Gamma^R \left[\Phi^R, h_{\mu\nu}\right] = \Gamma \left[v^R\right] + \sum_{n=2}^{\infty} \frac{1}{n!} \Gamma^{(n)}_{12..n} \left[v^R, h^{\mu\nu}\right] \left(\Phi^R_1 - v^R\right) \ldots\ldots\ldots \left(\Phi^R_n - v^R\right)$$

$$\text{where } \Gamma^{(n)}_{12..n} \left[v^R, h^{\mu\nu}\right] = \left. \frac{\delta^n \Gamma^{(R)} \left[\Phi^R, h^{\mu\nu}\right]}{\delta \Phi^R(x_1) \ldots \delta\Phi^R(x_n)} \right|_{\Phi^R = v^R} \tag{3}$$

and

$$\Gamma^{(n)}_{12..n} \left[v^R, h^{\mu\nu}\right] = \sum_{m=0}^{\infty} \frac{1}{m!} v^{mR} \Gamma^{(n+m)}_{12..n} \left[v^R = 0, h^{\mu\nu}\right]$$

We have

$$\left.\frac{\delta\Gamma^R\,[\Phi^R,\ h^{\mu\nu}]}{\delta h^{\mu\nu}}\right|_{h^{\mu\nu}=\ 0} = \sum_{n=2}^{\infty}\frac{1}{n!}\left.\frac{\delta\Gamma_{12..n}[v^R,h_{\mu\nu}]}{\delta\ h^{\mu\nu}}\right|_{h^{\mu\nu}=\ 0}\times$$

$$\left.\left(\Phi^R_1 - v^R\right)\ldots\left(\Phi^R_n - v^R\right)\right|_{h_{\mu\nu}=\ 0}$$

(5)

The left hand side of the relation written above, is finite, thus

$$\sum_{n=2}^{\infty}\frac{1}{n!}\left.\frac{\delta\Gamma^{(n)}_{12..n}[v^R,h_{\mu\nu}]}{\delta\ h_{\mu\nu}}\right|_{h_{\mu\nu}=\ 0}\left.\left(\Phi^R_1 - v^R\right)\ldots\left(\Phi^R_n - v^R\right)\right|_{h_{\mu\nu}=\ 0}$$

is finite. Hence it implies that $\left.\frac{\delta\Gamma^{(n)}_{12..n}[v^R,h_{\mu\nu}]}{\delta\ h_{\mu\nu}}\right|_{h_{\mu\nu}=\ 0}$ are finite too. These are the proper vertices with the insertion of $\theta^{imp}_{\mu\nu}$ in the broken $\lambda\phi^4$ theory and we have shown that they are finite. On the basis of this result we can say that the finite improvement program is sufficient to make the energy momentum tensor finite in the broken $\lambda\phi^4$ theory as well.

We wish to point out that the analysis that we have carried out for the $\lambda\phi^4$ theory, can be repeated for any other renormalizable theory with spontaneously broken symmetry and the claim is that the same result would follow i.e. once the energy momentum tensor is made finite in the unbroken theory, the broken theory poses no problem in the sense that the Green's functions with the insertion of energy momentum tensor

in the broken theory do not give rise to any new divergences. Of course the scheme of obtaining finite energy momentum tensor would differ as we go from $\lambda\phi^4$ theory to any other theory e.g. Scalar QED or W-S model.

We shall be making use of this result while dealing with W-S model since it is simpler to deal with the unbroken model. As a side remark, we wish to comment that the method we would adopt to renormalize the energy momentum tensor in the unbroken W-S model can be applied to the broken model as well. We are restricting to the former case just for the sake of simplification.

4.3 Generalization of 'RG covariance' criterion to W-S model

4.3.1 Preliminaries

We shall work with the action given below:

$$S_o = \int dx \Big[-\frac{1}{4} G_{\mu\nu}G^{\mu\nu} - \frac{1}{4} F^i_{\mu\nu} F^{\mu\nu i} + \Big[\sum_i \bar{\psi}_i \, i\gamma^\mu \big(\vec{\partial}_\mu - ig_o \frac{\tau . A_\mu}{2}$$

$$- ig'_o \frac{Y}{2} B_\mu\big)\psi_i - \sum_i \bar{\psi}_i \, i \big(\overleftarrow{\partial}_\mu + ig_o \frac{\tau}{2} . A_\mu + ig'_o \frac{Y}{2} B_\mu\big)\gamma^\mu\psi_i\Big] +$$

$$\big(D_\mu\phi\big)^\dagger \big(D^\mu\phi\big) - m_o^2 \phi^\dagger\phi - \lambda_o(\phi^\dagger\phi)^2 - \frac{\xi_o}{2}(\partial . A^a)^2 - \frac{\alpha_o}{2} (\partial . B)^2$$

$$+ \partial_\mu \bar{C}^a \big(\delta_{ac}\partial^\mu - g_o \in^{abc} A^{\mu b}\big) C_c\Big]$$

where $F^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g_o \in^{ijk} A^j_\mu A^k_\nu \quad i=1,2,3$

$$G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

are the SU(2) and U(1) gauge field tensors respectively. We make independent choices for the left handed and right handed fermions. $\psi = \nu_{eL},\ e_L,\ e_R,\ u_L,\ u_R,\ d_L,\ d_R$

$$D_\mu \psi = \left(\partial_\mu - i g_o \frac{\tau . A_\mu}{2} - i g'_o \frac{Y}{2} B_\mu\right) \psi$$

$$D_\mu \phi = \left(\partial_\mu - i g_o \frac{\tau . A_\mu}{2} - i g'_o \frac{Y}{2} B_\mu\right) \phi$$

ϕ is the complex scalar doublet with the charge assignment

$$\phi = \begin{bmatrix} \phi^+ \\ \phi^o \end{bmatrix} \ ; \quad \text{the hypercharge } Y(\phi) = 1$$

We have neglected SU(2) x U(1) Yukawa couplings between scalars and fermions for the sake of simplification.

The Einstein energy momentum tensor is given by

$$\theta_{\mu\nu}(x) = \frac{2}{\sqrt{-g(x)}} \frac{\delta S}{\delta g^{\mu\nu}(x)} \Bigg|_{g^{\mu\nu}(x) = \eta^{\mu\nu}}$$

where S is the action obtained from the flat space action S_o by making the following substitutions: all ordinary derivatives $\longrightarrow$ covariant derivatives, $\int d^4x \longrightarrow \int \sqrt{-g(x)}\, d^4x$ and $\eta^{\mu\nu} \longrightarrow g^{\mu\nu}(x)$

$$\theta_{\mu\nu} = - \eta_{\mu\nu}\mathcal{L} - F^i_{\mu\alpha} F^{i\alpha}_{\nu} - G_{\mu\alpha} G^{\alpha}_{\nu} + \left[(D_\mu\phi)^\dagger(D_\nu\phi) + (D_\nu\phi)^\dagger(D_\mu\phi)\right]$$

$$+\left[\frac{i}{4}\bar{\psi}_i\gamma_\mu\left(\overrightarrow{\partial}_\nu - ig_o\frac{\tau}{2}.A_\nu - ig'_o\frac{Y}{2}B_\nu\right)\psi_i - \frac{i}{4}\bar{\psi}_i\left(\overleftarrow{\partial}_\nu + ig_o\frac{\tau}{2}.A_\nu + ig'_o\frac{Y}{2}B_\nu\right)\gamma_\mu\psi_i + \mu \longleftrightarrow \nu\right]$$

$$+ \left[\partial_\mu \bar{C}^a\left(\partial_\nu C_b - g_o \epsilon^{abc} A^b_\nu C_c\right) + \mu \longleftrightarrow \nu\right]$$

$$+ \xi_o\left[\partial_\mu(\partial.A)A^a_\nu + \partial_\nu(\partial.A)A^a_\mu\right] - \eta_{\mu\nu}\xi_o\partial^\rho(\partial.A^a)A^a_\rho$$

$$- \eta_{\mu\nu}\xi_o(\partial.A^a)^2 + \alpha_o\left[\partial_\mu(\partial.B)B_\nu + \partial_\nu(\partial.B)B_\mu\right]$$

$$- \eta_{\mu\nu}\alpha_o\partial^\rho(\partial.B)B_\rho - \eta_{\mu\nu}\alpha_o(\partial.B)^2 \qquad (6)$$

The various renormalization constants in MS scheme are given below:

$$\psi^{UR} = Z_2^{1/2}\psi^R \qquad \phi^{uR} = Z^{1/2}\phi^R$$

$$B^{UR}_\mu = Z^{1/2}B^R_\mu \qquad \lambda_o = \mu^\epsilon(\lambda Z_\lambda + \delta\lambda)$$

$$A^{aUR}_\mu = Z_3^{1/2}A^{aR}_\mu \qquad g'_o = \mu^{\epsilon/2}g'\hat{Z}_3^{-1/2}$$

$$g_o = \mu^{\epsilon/2}g\frac{Z_1}{Z_2Z_3^{1/2}}$$

The renormalization group quantities are defined below:

$$\beta^\lambda(\lambda,g,g',\epsilon) = \mu\frac{\partial\lambda}{\partial\mu}\Big|_{\text{bare quantities fixed}}$$

$$= -\lambda\epsilon + \lambda^2\frac{\partial}{\partial\lambda}Z_\lambda^{(1)} + \frac{\lambda g}{2}\frac{\partial}{\partial g}Z_\lambda^{(1)} + \frac{\lambda g'}{2}\frac{\partial}{\partial g'}Z_\lambda^{(1)}$$

$$+ \frac{g}{2}\frac{\partial}{\partial g}(\delta\lambda)^{(1)} + \frac{g'}{2}\frac{\partial}{\delta g'}(\delta\lambda)^{(1)} - (\delta\lambda)^{(1)}$$

$$= -\lambda\epsilon + \tilde{\beta}_1^\lambda \lambda^2 + \beta_2^\lambda \lambda g'^2 + \beta_1^\lambda g'^4 + \vec{\beta}_3^\lambda g^4 + \vec{\beta}_4^\lambda g^4 g'^2 + \vec{\beta}_5^\lambda \lambda g^2$$

$$+ \ldots\ldots\ldots$$

$$\beta^g(\lambda, g, g', \epsilon) = \mu \frac{\partial g}{\partial \mu}\Big|_{\text{bare quantities fixed}}$$

$$= -\frac{g\epsilon}{2} + \lambda g \frac{\partial}{\partial \lambda} Z_g^{(1)} + \frac{g^2}{2}\frac{\partial}{\partial g} Z_g^{(1)} + \frac{gg'}{2}\frac{\partial}{\partial g'} Z_g^{(1)}$$

$$= -\frac{g\epsilon}{2} + \beta_3^g g^3 + \ldots\ldots\ldots$$

$$\beta^{g'}(\lambda, g^2, g'^2, \epsilon) = \frac{\mu\partial g'}{\partial\mu}\Big|_{\substack{\text{bare quantities}\\ \text{fixed}}}$$

$$= -\frac{g'\epsilon}{2} + \lambda g' \frac{\partial}{\partial \lambda} Z_{g'}^{(1)} \quad \frac{g'^2}{2}\frac{\partial}{\partial g'} Z_{g'}^{(1)} + \frac{gg'}{2}\frac{\partial}{\partial g'} Z_{g'}^{(1)}$$

$$= -\frac{g'\epsilon}{2} + \beta_3^{g'} g'^3 + \ldots\ldots\ldots$$

$$-\frac{\mu}{2}\frac{\partial}{\partial\mu} \operatorname{Ln} Z_m = \gamma_m(\lambda, g^2, g'^2) = \gamma_{m1}\lambda + \bar{\gamma}_{m1} g'^2 + \bar{\gamma}_{m2} g^2 + \ldots\ldots\ldots$$

$$\mu\frac{\partial\xi}{\partial\mu} = \gamma_\xi(\lambda, g^2, g'^2, \xi)\xi, \quad \mu\frac{\partial\alpha}{\partial\mu} = \gamma_\alpha(\lambda, g^2, g'^2)\alpha$$

$$Z_3^{-1}\mu\frac{\partial}{\partial\mu}Z_3 = \gamma_3(\lambda, g^2, g'^2, \xi), \quad \hat{Z}_3^{-1}\mu\frac{\partial}{\partial\mu}\hat{Z}_3 = \hat{\gamma}_3(\lambda, g^2, g'^2)$$

$$Z^{-1}\mu\frac{\partial}{\partial\mu}Z = \gamma(\lambda, g^2, g'^2, \alpha, \xi), \quad Z_2^{-1}\mu\frac{\partial}{\partial\mu}Z_2 = \gamma_\psi(\lambda, g^2, g'^2, \xi, \alpha)$$

The values of the renormalization group quantities calculated

explicitly and which will be used in section 4.3.3 and 4.3.5 are given below:

$$\gamma_{m1} = \frac{1}{12\Pi^2} \qquad \tilde{\beta}_1^{\lambda} = \frac{1}{2\Pi^2}$$

$$\bar{\gamma}_{m1} = \frac{-3}{64\Pi^2} \qquad \beta_1^{\lambda} = \frac{3}{256\Pi^2}$$

$$\bar{\gamma}_{m2} = -\frac{9}{64\Pi^2} \qquad \beta_2^{\lambda} = -\frac{1}{4\Pi^2}$$

$$\beta_3^{g} = -\frac{39}{96\Pi^2} \qquad \vec{\beta}_3^{\lambda} = \frac{3}{256\Pi^2}$$

$$\beta_3^{g'} = \frac{31}{288\Pi^2} \qquad \vec{\beta}_4^{\lambda} = \frac{3}{128\Pi^2}$$

$$\vec{\beta}_5^{\lambda} = -\frac{1}{\Pi^2}$$

4.3.2 RG equation satisfied by $\langle\theta_{\mu\nu}^{imp}\rangle$ and RG condition on $\bar{k}$

To renormalize the energy momentum tensor in W-S model given by equation (6), we adopt the same approach as we did for the scalar QED. We add to $\theta_{\mu\nu}$, an improvement term with the most general coefficient. The improved energy momentum tensor is given by

$$\theta_{\mu\nu}^{imp} = \theta_{\mu\nu} - H_o\,(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\,(\phi^\dagger\phi)$$

where $H_o = G(\lambda, g^2, g'^2, \xi, \alpha, \epsilon) + \left(k(\lambda, g^2, g'^2, \frac{m^2}{\mu^2}, \xi, \alpha, \epsilon) + h\right) Z_m$

G contains only the pole terms. The symbols k, h and Z_m have

been defined already in the context of scalar QED. Such a procedure of obtaining a finite energy momentum tensor amounts to infinite renormalization of $\theta_{\mu\nu}$. The improved energy momentum tensor is finite but not unique. We fix it uniquely with the help of 'RG covariance' criterion, which will be taken up shortly.

We start with the derivation of the RG equation for $\Gamma_{\theta_{\mu\nu}}$ (the unrenormalized proper vertices of $\theta^{imp}_{\mu\nu}$). We consider the following expression:

$$\Gamma_{\theta_{\mu\nu}} = \Gamma^{(1)}_{\theta_{\mu\nu}} - H_o\,(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\,\Gamma^{(2)}$$

where $\Gamma^{(1)}_{\theta_{\mu\nu}}$ and $\Gamma^{(2)}$ are the unrenormalized proper vertices of $\theta_{\mu\nu}$ and $\phi^{\dagger}\phi$ respectively.

Consider for concreteness sake, a proper vertex with p_1 U(1) gauge field lines, n SU(2) gauge field lines, r fermion lines and 2q scalar lines. Then following the same steps as in scalar QED, we obtain

$$\Big\{\mu\frac{\partial}{\partial\mu} + \beta^{\lambda}(\lambda,g^2,g'^2,\epsilon)\frac{\partial}{\partial\lambda} + \beta^{g}(\lambda,g^2,g'^2,\epsilon)\frac{\partial}{\partial g} + \beta^{g'}(\lambda,g^2,g'^2,\epsilon)\frac{\partial}{\partial g'}$$
$$+ \xi\gamma_\xi\frac{\partial}{\partial\xi} + \alpha\,\gamma_\alpha\frac{\partial}{\partial\alpha} + 2\gamma_m m^2\frac{\partial}{\partial m^2} + \left(\frac{p}{2}\hat{\gamma}_3 + q\gamma + \frac{n}{2}\gamma_3 + \frac{r}{2}\gamma_\psi\right)$$
$$+ \delta\left(\lambda,g^2,g'^2,\frac{m^2}{\mu^2},\epsilon\right)\frac{\partial}{\partial h}\Big\}\,\Gamma^{R(p,n,r,2q)}_{\theta_{\mu\nu}} = 0 \qquad (7)$$

with

$$\delta\left[\lambda,g^2,g'^2,\frac{m^2}{\mu^2},\epsilon\right] = -Z_m^{-1}\left[\beta^\lambda(\lambda,g^2,g'^2,\epsilon)\frac{\partial}{\partial\lambda} + \beta^g(\lambda,g^2,g'^2,\epsilon)\frac{\partial}{\partial g'}\right.$$

$$\left. + \beta^{g'}(\lambda,g^2,g'^2,\epsilon)\frac{\partial}{\partial g'}\right]G + 2(h+k)\gamma_m - \left[\beta^\lambda(\lambda,g^2,g'^2,\epsilon)\frac{\partial}{\partial\lambda} + \right.$$

$$\left.\beta^g(\lambda,g^2,g'^2,\epsilon)\frac{\partial}{\partial g} + \beta^{g'}(\lambda,g^2,g'^2,\epsilon)\frac{\partial}{\partial g'} + (2\gamma_m - 2)m^2\frac{\partial}{\partial m^2}\right]k$$

(The physical matrix elements of $\theta_{\mu\nu}^{imp}$ should be independent of gauge parameters ξ and α. It requires that k be independent of ξ and α as is shown in appendix E.2)

The above equation (7) implies that the proper vertices of $\theta_{\mu\nu}^{imp}$ are 'RG covariant' iff the coefficient of the term $\frac{\partial}{\partial h}\Gamma_{\theta_{\mu\nu}}$ vanishes. This requires that

$$-\bar{\delta} = 0 = \left[\bar{\beta}^\lambda\frac{\partial}{\partial\lambda} + \bar{\beta}^g\frac{\partial}{\partial g} + \bar{\beta}^{g'}\frac{\partial}{\partial g'} + (2\gamma_m - 2)m^2\frac{\partial}{\partial m^2}\right]\bar{k}$$

$$-2\bar{k}\,\gamma_m - \left[\lambda\frac{\partial}{\partial\lambda} + \frac{g}{2}\frac{\partial}{\partial g} + \frac{g'}{2}\frac{\partial}{\partial g'}\right]G^{(1)}(\lambda,g^2,g'^2) = 0 \qquad (8)$$

where $\bar{k}$ = k+h and the rest of the bar quantities are defined for $\epsilon = 0$.

$G^{(1)}$ are the coefficients of simple pole terms in $G(\lambda,g^2,g'^2\epsilon)$ and we have made use of the fact that h is independent of $\lambda,g^2,g'^2,m,\epsilon$.

Our aim is to look for the solutions of equation (8). It is a single differential condition on a function of three variables and when a certain boundary condition is imposed, it can

either have a unique solution, no solution or infinitely many solutions. If a unique solution exists, it will fix the quantity (k+h) appearing in $\theta^{imp}_{\mu\nu}$, choosing a particular energy momentum tensor of a set of infinite number of them. We wish to investigate physically meaningful boundary conditions that yield a unique solution for Eq. 8.

4.3.3 Solution perturbative in λ, g^2, g'^2

We impose the boundary condition on $\bar{k}(\lambda,\ g^2,\ g'^2,\ \frac{m^2}{\mu^2})$ that it should have a perturbative expansion in powers of λ, g^2, g'^2 and it should have a finite limit as $m \longrightarrow 0$. Using the second part of the boundary condition we can simplify equation (8) to

$$\left(\bar{\beta}^{\lambda} \frac{\partial}{\partial\lambda} + \bar{\beta}^{g} \frac{\partial}{\partial g} + \bar{\beta}^{g'} \frac{\partial}{\partial g'} - 2\gamma_m\right) \bar{k}(\lambda,\ g^2,\ g'^2) =$$

$$\left(\lambda \frac{\partial}{\partial\lambda} + g^2 \frac{\partial}{\partial g^2} + g'^2 \frac{\partial}{\partial g'^2}\right) G^{(1)}(\lambda,\ g^2,\ g'^2) \tag{9}$$

Since the pole parts in $G(\lambda,\ g^2,\ g'^2\ \epsilon)$ are related to those in Z_m^{-1} via the relation

$$p.p\left[\ (G + \bar{k}\ Z_m)\ Z_m^{-1}\ 2(1-n) + (n-2)\ Z_m^{-1}\ \right]$$

$$\equiv p.p\left[2 \times Z_m^{-1}\ (\epsilon-3) + (2-\epsilon)\ Z_m^{-1}\ \right] = 0 \tag{10}$$

upto $O(\lambda^3,\ \lambda g^2,\ \lambda g'^2,\ g^4,\ g'^4,\ g^2 g'^2)$, as shown in the appendix E.1, we rewrite equation (9) in terms of X.

We get

$$\left[\left(\bar{\beta}^{\lambda} - \lambda\epsilon\right)\frac{\partial}{\partial\lambda} + \left(\bar{\beta}^{g} - \frac{g\epsilon}{2}\right)\frac{\partial}{\partial g} + \left(\bar{\beta}^{g'} - \frac{g'\epsilon}{2}\right)\frac{\partial}{\partial g'}\right] x + \left(\lambda\epsilon \frac{\partial}{\partial\lambda} \bar{k}\right) Z_m$$

$$+ \frac{g\epsilon}{2}\left(\frac{\partial}{\partial g}\bar{k}\right) Z_m + \frac{g'\epsilon}{2}\left(\frac{\partial}{\partial g'}\bar{k}\right) Z_m = 0 \qquad (11)$$

To derive the above relation, we keep in mind that

$Z_m^{-1}\left[\beta^{\lambda}\frac{\partial}{\partial\lambda} + \beta^{g'}\frac{\partial}{\partial g'} + \beta^{g}\frac{\partial}{\partial g}\right] G$ is finite because δ is finite and G has only poles.

In $O(\epsilon^{o})$ the above equation reduces to

$$-\frac{\lambda\partial}{\partial\lambda}X^{(1)} - \frac{g}{2}\frac{\partial}{\partial g}X^{(1)} - \frac{g'}{2}\frac{\partial}{\partial g'}X^{(1)} + \left[\lambda\frac{\partial}{\partial\lambda}\bar{k}\right] Z_m^{(1)} + \left[\frac{g}{2}\frac{\partial}{\partial g}\bar{k}\right] Z_m^{(1)}$$

$$+ \left[\frac{g'}{2}\frac{\partial}{\partial g'}\bar{k}\right] Z_m^{(1)} = 0 \qquad (12)$$

We wish to check whether the perturbative solution exists for this equation. For that we expand

$$X = X^{(o)} + \frac{Y^{(1)}\lambda}{\epsilon} + \frac{\bar{X}^{(1)}g^{2}}{\epsilon} + \frac{\tilde{X}^{(1)}g'^{2}}{\epsilon} + \frac{\hat{X}^{(1)}\lambda^{2}}{\epsilon} +$$

$$\frac{X_o^{(1)}\lambda g^{2}}{\epsilon} + \frac{X_1^{(1)}\lambda g'^{2}}{\epsilon} + \frac{X_2^{(1)}g'^{4}}{\epsilon} + \frac{X_3^{(1)}g^{4}}{\epsilon} + \frac{X_4^{(1)}g^{2}g'^{2}}{\epsilon} + \ldots$$

$$Z_m^{-1} = 1 + \frac{a\lambda}{\epsilon} + \frac{bg'^{2}}{\epsilon} + \frac{cg^{2}}{\epsilon} + \frac{d\,\lambda g^{2}}{\epsilon} + \frac{\bar{d}\,\lambda g'^{2}}{\epsilon} + \ldots\ldots\ldots$$

$$\bar{k} = \bar{k}_o + \bar{k}_1\lambda + \bar{k}_2 g^{2} + \bar{k}_3 g'^{2} + \ldots\ldots\ldots$$

Considering equation (12) in various orders in coupling constants gives six equations of constraint on $\bar{k}_1$, $\bar{k}_2$ and $\bar{k}_3$ which are given below:

In $O(\lambda^2)$ $-\bar{k}_1 a - 2\hat{X}^{(1)} = 0$ (13i)

$O(\lambda g'^2)$ gives $-2X_1^{(1)} - \bar{k}_1 b - \bar{k}_3 a = 0$ (13ii)

$O(g'^4)$ gives $-2X_2^{(1)} - \bar{k}_3 b = 0$ (13iii)

$O(\lambda g^2)$ gives $-2X_o^{(1)} - \bar{k}_1 c - \bar{k}_2 a = 0$ (13iv)

$O(g^4)$ gives $-2X_3^{(1)} - \bar{k}_2 c = 0$ (13v)

$O(g^2 g'^2)$ gives $-2X_4^{(1)} - \bar{k}_2 b - \bar{k}_3 c = 0$ (13vi)

The equations written above are inconsistent even for g=0. To prove the inconsistency, we take the first three equations. With the help of equations (10) and (11), $\hat{X}^{(1)}$, $X_1^{(1)}$ and $X_2^{(1)}$ can all be related to terms in Z_m^{-1}. Thus we get a relation in terms of calculable quantities, which must be satisfied if these equations have to be consistent. The relation is as follows:

$$\beta_2^\lambda a - \tilde{\beta}_1^\lambda b - \beta_1^\lambda \frac{a^2}{b} - 2\beta_3^{g'} a = 0.$$

With the values given in section 4.3.1, it is verified that this condition is not met. Thus the obvious perturbative boundary condition on $\bar{k}$ gives no solution to equation (12).

4.3.4 Solution perturbative around a nontrivial fixed point λ^*, g^{*2}, $g'^{*2} \neq 0$.

We assume the existence of a nontrivial fixed point λ^*, $g^{*2}, g'^{*2} \neq 0$ in the theory. We require that $\bar{k}(\lambda, g^2, g'^2, \frac{m^2}{\mu^2})$ be perturbative in powers of $(\lambda-\lambda^*)$, (g^2-g^{*2}) and $(g'^2-g'^{*2})$. We also impose the condition that $\bar{k}$ be analytic in m^2. First we consider the part of $\bar{k}$ that is independent of $m, \bar{k}(\lambda, g^2, g'^2) \equiv \bar{k}(\lambda, g^2, g'^2, m=0)$

We expand

$$\bar{k}(\lambda, g^2, g'^2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \bar{k}_{mnl}\,(\lambda-\lambda^*)^m\,(g'^2-g'^{*2})^n(g^2-g^{*2})^l$$

$$\bar{\beta}^\lambda(\lambda, g^2, g'^2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \beta_{mnl}(\lambda-\lambda^*)^m(g'^2-g'^{*2})^n(g^2-g^{*2})^l$$

$$2g'\,\bar{\beta}^{g'}(\lambda, g^2, g'^2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \bar{\beta}^{g'}_{mnl}(\lambda-\lambda^*)^m(g'^2-g'^{*2})^n(g^2-g^{*2})^l$$

$$2g\,\bar{\beta}^{g}(\lambda, g^2, g'^2) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum_{l=0}^{\infty} \bar{\beta}_{mnl}(\lambda-\lambda^*)^m\,(g'^2-g'^{*2})^n(g^2-g^{*2})^l$$

$$\gamma^m(\lambda, g^2, g'^2) = \sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty} \gamma^m_{pqr}\,(\lambda-\lambda^*)^p\,(g'^2-g'^{*2})^q(g^2-g^{*2})^r$$

and

$$\left[\lambda\frac{\partial}{\partial\lambda} + g'^2\frac{\partial}{\partial g'^2} + g^2\,\frac{\partial}{\partial g^2}\right] G^{(1)}(\lambda, g^2, g'^2) = \xi(\lambda, g^2, g'^2)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \xi_{mnp} (\lambda-\lambda^*)^m (g^2 - g^{*2})^n (g^2 - g^{*2})^p$$

We consider the equation

$$\left[\bar{\beta}^{\lambda} \frac{\partial}{\partial\lambda} + 2g'\bar{\beta}^{g'}\frac{\partial}{\partial g'} + 2g\,\bar{\beta}^{g} \frac{\partial}{\partial g} - 2\gamma_m\right] \bar{k}\,(\lambda, g^2,\ g'^2) = \xi(\lambda, g^2,\ g'^2) \tag{14}$$

in $0(\lambda-\lambda^*)^o (g'^2 - g'^{*2})^o\ (g^2 - g^{*2})$ first. One gets

$$\bar{k}_{ooo} = -\frac{1}{2}\frac{\xi_{ooo}}{\gamma^m_{ooo}}, \quad \text{assuming } \gamma^m_{ooo} = \gamma_m(\lambda^*) \neq 0$$

We proceed by induction now. Assume $\bar{k}_{mnl}$ have been fixed uniquely by equation (14) for $m+n+l \leq N$ where $0 \leq m,n,l \leq N+1$. Then it is easily shown that $\bar{k}_{mnl}$ with $m+n+l = N+1$, are fixed via equation (14). Thus $\bar{k}\ (\lambda, g^2, g'^2)$ is fixed uniquely in this case.

Next we consider the mass dependent part of $\bar{k}\left(\lambda, g^2, g'^2, \frac{m^2}{\mu^2}\right)$. We write $\bar{k}(\lambda, g^2, g'^2, \frac{m^2}{\mu^2}) = \bar{k}(\lambda, g^2, g'^2) + \sum_{n=1}^{\infty} \left(\frac{m^2}{\mu^2}\right)^n\ \bar{k}_n(\lambda, g^2, g'^2)$.

Using equations (8) and (9), we get

$$\bar{\beta}^{\lambda} \frac{\partial}{\partial\lambda} + \bar{\beta}^{g'}\frac{\partial}{\partial g} + \bar{\beta}^{g'} \frac{\partial}{\partial g'} + \left[(2n-2)\gamma_m - 2n\right] \bar{k} = 0 \text{ for } n \geq 1$$

It is easy to see that the only perturbative (in $(\lambda-\lambda^*)$ $(g^2 - g^{*2})$ $(g'^2 - g'^{*2})$) solution to the above equation is

$$\bar{k}_n(\lambda, g^2, g'^2) = 0 \quad \text{for } n \geq 1.$$

Hence the boundary conditions considered here,

fix $\bar{k}_n(\lambda, g^2, g'^2 \frac{m^2}{\mu^2})$ uniquely. $\bar{k}_n(\lambda, g^2, g'^2 \frac{m^2}{\mu^2})$ has analytic behavior about the fixed point, we considered, thus $\theta^{imp}_{\mu\nu}$ (which is now fixed uniquely) has analytic behavior about this fixed point.

4.3.5 Less restrictive 'RG covariance' condition

We have already shown that the perturbative boundary condition gives no solution to equation (12). We recollect that this equation is obtained when $\bar{\delta}$ is set equal to zero in all orders in coupling constants.

We now consider a condition some what less restrictive than 'RG covariance' condition and which gives a perturbative solution to equation (12). We consider the following equation:

$$- \lambda \frac{\partial}{\partial\lambda} X^{(1)} - \frac{g}{2} \frac{\partial}{\partial g} X^{(1)} - \frac{g'}{2} \frac{\partial}{\partial g'} X^{(1)} + (\lambda \frac{\partial}{\partial\lambda} \bar{k}) Z_m^{(1)} + (\frac{g}{2} \frac{\partial}{\partial g} \bar{k}) Z_m^{(1)} + (\frac{g'}{2} \frac{\partial}{\partial g'} \bar{k}) Z_m^{(1)} = - \bar{\delta} \qquad (15)$$

As we had shown for scalar QED, in W-S model also, it is possible to choose $\bar{k}n$'s appropriately in the constraint equations such that $\bar{\delta} = 0$ in $O(\lambda, g^2, g'^2)$ and $O(\lambda^{n+1}, \lambda^n (g^2)^p (g'^2)^q)$ [where n runs from 1 to ∞ and $p+q = N$, $0 \leq p, q \leq N$ and N runs from 1 to

∞] ,as shown in Appendix F. This fixes $\bar{k}$ $(\lambda,\ g^2,\ g'^2)$ uniquely and hence the $\theta^{imp}_{\mu\nu}$ is fixed uniquely. The first non zero contribution to $\bar{\delta}$ comes in $O(g^4, g'^4, g^2g'^2)$ and which we have calculated explicitly by considering equations (13i), (13ii), (13iv) and equation (15) in $O(g^4, g'^4, g^2g'^2)$and using the values of different calculable quantities. The physical significance of $\bar{\delta}$ not being zero can be understood, as explained in the context of scalar QED, provided $\lambda \gg g^2$ and g'^2, but small enough so that perturbative expansion can be carried out in λ. Under these assumptions, the contribution of the term $\bar{\delta}\,\frac{\partial}{\partial h}\,\Gamma_{\theta_{\mu\nu}}$, is very small as compared to the typical term $\bar{\beta}^{\lambda}\,\frac{\partial}{\partial \lambda}\,\Gamma_{\theta_{\mu\nu}}$, in the RG equation satisfied by $\Gamma_{\theta_{\mu\nu}}$.

Conclusions:

In Weinberg-Salam model also, it is possible to obtain a unique, finite energy momentum tensor when `RG covariance' criterion together with certain boundary conditions are applied at a non trivial fixed point $\lambda^*$, $g'^{*2}$, $g^{*2}$ ($\neq 0$) of the theory. We have shown that `RG covariance' criterion when taken perturbatively in λ, g^2, g'^2 did not yield any unique energy momentum tensor. A criterion somewhat weaker than `RG covariance' criterion, according to which $\bar{\delta}$ is not made zero in all orders λ, g^2, g'^2 but only in selected orders as shown in the last section, did give rise to a unique, finite $\theta^{imp}_{\nu\sigma}$.

CHAPTER - 5

TRACE ANOMALY NEAR THE FIXED POINT IN $\lambda\phi^4$ THEORY

5.1 Introduction

In a massless classical field theory involving only dimensionless parameters, it is possible to define an energy momentum tensor with vanishing trace. In general, in the corresponding quantized theory, the same energy momentum tensor does not have a vanishing trace. This non vanishing trace which appears in the quantized theory is known as trace anomaly.

In a specific quantum field theory ($\lambda\phi^4$ model), Schroer[19] studied the trace anomaly of the renormalized energy momentum tensor (expressed in terms of normal products). He found that the trace anomaly of this energy momentum tensor could be soft (which means that it should contain operators of dimension less than four). The necessary and sufficient condition for the softness of the trace anomaly is the vanishing of the beta function.

In the work presented here, we shall examine Schroer's criterion in the context of $\lambda\phi^4$ theory, in which the infinities of Green's functions are regularized using dimensional regularization. We shall be dealing with the trace anomaly of the energy momentum tensor whose certain unique properties will be discussed later. It was constructed by Collins [8] and has the

form $\theta_{\mu\nu}^{imp} = \partial_\mu\phi\ \partial_\nu\phi - \eta_{\mu\nu}\mathcal{L} + H_o(\epsilon)\ (\partial_\mu\partial_\nu - \eta_{\mu\nu}\ \partial^2)\phi^2$ (5.1)

(where $H_o(\epsilon)$ is a unique series in $\epsilon = 4-n$ with only non negative powers of ϵ and is determined successively in each order in perturbation theory). This energy momentum tensor is finite to all orders in perturbation theory.

Let us briefly go through the properties of the energy momentum tensor mentioned above,as they have already been taken up in chapter 1. It was shown by Collins that the above $\theta_{\mu\nu}^{imp}$ is the only energy momentum tensor of the form,

$$\theta_{\mu\nu}^{imp} = \partial_\mu\phi\ \partial_\nu\phi - g_{\mu\nu}\mathcal{L} + H_o(\epsilon,\ \lambda,\ \frac{m^2}{\mu^2})\ (\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2 \qquad (5.2)$$

(where $H_o(\epsilon, \lambda,\ \frac{m^2}{\mu^2})$ is a finite function of renormalized parameters λ and m at $\epsilon = 0$), that is finite to all orders in perturbation theory. Energy momentum tensor of eqn (5.1) is also unique energy momentum tensor of the form [13]

$$\theta_{\mu\nu}^{imp} = \partial_\mu\phi\partial_\nu\phi - \eta_{\mu\nu}\mathcal{L} + H_o(\epsilon, \lambda_o\mu^{-\epsilon}, \frac{m_o^2}{\mu^2})(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2 \qquad (5.3)$$

(where $H_o(\epsilon, \lambda_o\mu^{-\epsilon},\ \frac{m_o^2}{\mu^2})$ is a finite function of bare parameters $\lambda_o, \frac{m_o^2}{\mu^2}$ at $\epsilon = 0$). Here μ is the arbitrary parameter used in dimensional regularization. In the two cases given by equations (5.2) and (5.3), the canonical energy momentum tensor $\theta^{\mu\nu} = \partial_\mu\phi\ \partial_\nu\phi - \eta_{\mu\nu}\mathcal{L}$ has been improved by adding $H_o(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2$ improvement terms to obtain energy momentum tensor that is finite,

but there is a difference in these cases. In the first case H_o is a finite function of renormalized parameter λ and m at $\epsilon = 0$ whereas in the second case H_o is a finite function of parameters λ_o, $\frac{m_o^2}{\mu^2}$ at $\epsilon = 0$. This scheme of improving $\theta^{\mu\nu}$ to get finite $\theta^{\mu\nu\,imp}$ employing either of the two H_o's in the improvement term is called the finite improvement program. The advantage of $\theta^{imp}_{\mu\nu}$ of equation (5.3) over that of equation (5.2) is that it is derivable from the action that is a finite function of bare quantities and is, as a whole, a finite function of bare quantities. As is discussed in chapter 2 in detail, if it is possible to obtain finite energy momentum tensor using finite improvement program, the flat space parameters are sufficient to specify the theory completely in the presence of gravity.

Thus the energy momentum tensor $\theta^{imp}_{\mu\nu}$, we shall consider, has a unique significance outlined above and it has a non trivial trace. Collins [8] has argued that this trace anomaly is soft at the fixed point of the theory and thus consistent with Schroer's result. The main purpose of this work is to show that in general, when the theory has a non trivial fixed point, it may not be true [21].

5.2 Preliminaries

Let us go through the derivation of the trace anomaly, we shall be working with, in brief. We shall work in the context of $\lambda\phi^4$ theory whose Lagrange density is given by

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi - \frac{1}{2}\,m_o^2\phi^2 - \frac{\lambda_o}{4!}\,\phi^4 \;;\; S = \int d^n x\,\mathcal{L} \qquad (5.4)$$

We shall use minimal subtraction scheme (MS scheme) throughout to determine the renormalization constants for $\mathcal{L}$ and for operators. The renormalization transformations are

$$\phi^{UR} = Z^{1/2}\,\phi^{R} \qquad m_o{}^2 = Z_m\, m^2 \qquad \lambda_o = \mu^{\varepsilon}\lambda\, Z_\lambda \tag{5.5}$$

Here Z, Z_m, Z_λ are independent of m in MS scheme.

The following set of operators [8,11] is closed under renormalization:

$$\begin{aligned} O_1 &= \phi\,(\partial^2 + m_o{}^2)\,\phi & \qquad O_2 &= \frac{\delta s}{\delta\phi}\,\phi \\ O_3 &= m_o{}^2\,\phi^2 & O_4 &= \partial^2\phi^2 \end{aligned} \tag{5.6}$$

O_2 and O_3 are finite operators and O_4 is multiplicatively renormalized [8,11]:

$$\left\{\partial^2\phi^2\right\}^{U.R.} = Z_m^{-1}\left\{\partial^2\phi^2\right\}^{R} \tag{5.7}$$

We define the renormalization matrix by

$$\left\{O_i\right\}^{U.R} = \sum_j Z_{ij}\left\{O_j\right\}^{R} \tag{5.8}$$

As shown in References [8,11] Z_{ij} has the structure

$$Z_{ij} = \begin{bmatrix} 1 - \dfrac{\beta(\lambda}{\lambda\varepsilon} & -4\left(\dfrac{\beta(\lambda)}{4\lambda\varepsilon} - \dfrac{\gamma(\lambda)}{\varepsilon}\right) & \dfrac{4\gamma_m(\lambda)}{\varepsilon} & Z_{14} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & Z_m^{-1} \end{bmatrix}. \tag{5.9}$$

where $\beta(\lambda)$, $\gamma(\lambda)$, $\gamma_m(\lambda)$ have been defined by the standard renormalization group definitions:

$$\beta(\lambda,\epsilon) = -\lambda\epsilon + \beta(\lambda) = \mu \frac{\partial\lambda}{\partial\mu}\Bigg|_{\lambda_o, m_o, \epsilon}$$

$$\gamma(\lambda) = \mu \frac{\partial}{\partial\mu} \ln Z \Bigg|_{\lambda_o, m_o, \epsilon}$$

$$\gamma_m(\lambda) = -\frac{1}{2}\mu \frac{\partial}{\partial\mu} \ln Z_m \Bigg|_{\lambda_o, m_o, \epsilon} \tag{5.10}$$

As Collins has shown, the following energy-momentum tensor has finite matrix elements to all orders in λ :

$$\theta_{\mu\nu}^{imp} = \partial_\mu\phi\, \partial_\nu\phi - \eta_{\mu\nu}\mathcal{L} + \frac{g(\epsilon)}{1-n}(\partial_\mu\partial_\nu - \partial^2\eta_{\mu\nu})\phi^2 \tag{5.11}$$

where $g(\epsilon)$ is a series in nonnegative powers of $\epsilon = 4-n$. We redefine

$$g(\epsilon) = \frac{n-2}{4} - \tilde{g}(\epsilon)\frac{\epsilon}{4} \tag{5.12}$$

Then it can be shown that $\tilde{g}(\epsilon)$ begins as $O(\epsilon^2)$.

It is easily shown that [8,11]

$$\langle\theta_\mu^{\mu\, imp}\rangle = -\frac{\epsilon}{4}\langle O_1\rangle^{U.R.} - \frac{n}{4}\langle O_2\rangle^R + \langle O_3\rangle^R - \frac{\epsilon}{4}\tilde{g}(\epsilon) Z_m^{-1}\langle O_4\rangle^R \tag{5.13}$$

or, using the expressions for Z_{ij} ($j=1,2,3$), this becomes

$$\langle\theta_\mu^{\mu\, imp}\rangle = \frac{\beta(\lambda)}{4\lambda}\langle O_1\rangle^R + \left(\frac{\beta(\lambda)}{4\lambda} - \gamma\right)\langle O_2\rangle^R - \gamma_m\langle O_3\rangle^R$$

$$- \frac{1}{4} X^{(1)} \langle 0_4 \rangle^R - \langle 0_2 \rangle^R + \langle 0_3 \rangle^R + 0(\epsilon) \tag{5.14}$$

where

$$X = Z_{14} + \tilde{g}(\epsilon) Z_m^{-1} \tag{5.15}$$

and $X^{(1)}$ denotes the coefficient of simple pole terms in X; viz.

$$X^{(1)} = Z_{14}^{(1)} + K(\lambda) \tag{5.16}$$

$K(\lambda)$ being the coefficient of simple pole in $\tilde{g}(\epsilon) Z_m^{-1}$.

It can be shown that Z_{14} satisfies the RG equations

$$\frac{\mu \partial}{\partial \mu} Z_{14} = \left[-\lambda\epsilon + \beta(\lambda)\right] \frac{\partial Z_{14}}{\partial \lambda} = 2 \gamma_m Z_{14} + \gamma_{14} Z_{11} \tag{5.17}$$

and consequently X satisfies

$$\mu \frac{\partial}{\partial \mu} X = [-\lambda\epsilon + \beta(\lambda)] \frac{\partial X}{\partial \lambda} = 2 \gamma_m X + \gamma_{14} Z_{11} \tag{5.18}$$

Eq. (5.17) implies that

$$\gamma_{14}(\lambda) = - \lambda \frac{\partial}{\partial \lambda} Z_{14}^{(1)} \tag{5.19}$$

$\tilde{g}(\epsilon)$ is chosen in perturbation theory requiring that X has no worse than simple poles. Then Eq. (5.18) implies that

$$\beta(\lambda) \frac{\partial}{\partial \lambda} K(\lambda) = 2\gamma_m X^{(1)} = 2\gamma_m(\lambda) \left[Z_{14}^{(1)}(\lambda) + K(\lambda) \right] \tag{5.20}$$

$K(\lambda)$ is chosen successively in perturbation series from Eq.(5.20) and since terms in $K(\lambda)$ and $\tilde{g}(\epsilon)$ are related in a one-one manner, Eq.(5.20) fixes $\tilde{g}(\epsilon)$ uniquely in perturbation series, given $Z_{14}^{(1)}$.

The anomalous part of $\langle\theta_\mu^{\mu\,imp}\rangle$ is then [See Eq.(5.14)]

$$\langle\theta_\mu^{\mu\,imp}\rangle_{anan} = \frac{\beta(\lambda)}{4\lambda}\langle O_1\rangle^R + \left[\frac{\beta(\lambda)}{4\lambda} - \gamma\right]\langle O_2\rangle^R - \gamma_m\langle O_3\rangle^R$$

$$- \frac{1}{4}\left[Z_{14}^{(1)} + K(\lambda)\right]\langle O_4\rangle^R \tag{5.21}$$

$$= \frac{\beta(\lambda)}{4\lambda}\langle O_1\rangle^R + \left[\frac{\beta(\lambda)}{4\lambda} - \gamma\right]\langle O_2\rangle^R - \gamma_m\langle O_3\rangle^R$$

$$- \frac{1}{8}\frac{\beta(\lambda)}{\gamma_m(\lambda)}\frac{d}{d\lambda}K(\lambda)\langle O_4\rangle^R \tag{5.22}$$

The $\theta_\mu^{\mu\,imp}$ in Eqn. (5.22) is valid to all orders in perturbation theory.

Green's functions $\langle O_2\rangle^R$ vanish on-shell at nonzero momentum q. In the massless limit $\langle O_3\rangle^R = m^2\langle\phi^2\rangle^R$ also vanishes. Thus it would appear, on the face of it, that at a fixed point λ^* for which $\beta(\lambda^*) = 0$, the trace anomaly would always vanish as the coefficients of $\langle O_1\rangle^R$ and $\langle O_4\rangle^R$ are proportional to $\beta(\lambda)$. However as shown in the Sec.(5.3), this is not always true.

5.3 Expression for trace anomaly in the neighbourhood of a fixed point.

In this section we shall deal with the coefficient of $\langle O_4\rangle^R$ in anomaly equation (5.22). It should be emphasized that

as $K(\lambda)$ is not arbitrary but is restricted by the requirement that it should satisfy Eq. (5.20) which involves $\beta(\lambda)$, $K(\lambda)$ can show an unusual behavior near $\lambda = \lambda^*$. There is no reason to expect that like other quantities $K(\lambda)$ will be smooth near a fixed point λ^*. This is confirmed by the calculations below.

We shall do our calculations in the context of an ultraviolet stable fixed point $\lambda^* \neq 0$ (though the calculations can be done assuming an infrared stable fixed point which is nonzero). For an arbitrary mass scale $\mu (0<\mu<\infty)$, $\bar{\lambda}(\mu)$ will be restrictedto be below λ^* and then as $\mu \to \infty$, $\bar{\lambda}(\mu) \to \lambda^*$. We shall assume that $\beta(\lambda)$ is analytic at λ^* and for λ close to λ^* has the form given below:

$$\beta(\lambda) \approx a(\lambda^*-\lambda)^n \text{ (where } n \geq 1) \tag{5.23}$$

(The fact that λ^* is an ultraviolet stable fixed point and that it is being approached from below fixes `a' to be greater than zero.) Equation (5.20) has an exact solution:

$$K(\lambda) = \exp\left(-\int_{\lambda}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda')}{\beta(\lambda')} d\lambda'\right)\left[C_0 - \int_{\lambda}^{\lambda^*-\delta} d\lambda' \frac{2\gamma_m(\lambda')Z_{14}^{(1)}(\lambda')}{\beta(\lambda')} \times \right.$$

$$\left. \times \exp \int_{\lambda'}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda'')}{\beta(\lambda'')} d\lambda''\right] \tag{5.24}$$

where C_0 is the constant of integration and δ is an arbitrary small but positive number. C_0 is not arbitrary but is fixed uniquely when one uses the boundary condition on $K(\lambda)$, which is $K(0) = 0$. $K(\lambda)$ is a power series in λ and is chosen successively

in perturbation with the help of eqn.(5.20).

We consider equation (5.24) at $\lambda = 0$. The left hand side vanishes because of the boundary condition and we obtain,

$$C_o = \left[\int_0^{\lambda^*-\delta} d\lambda' \frac{2\gamma_m(\lambda')\, Z_{14}(\lambda')}{\beta(\lambda')} \exp \int_{\lambda'}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda'')}{\beta(\lambda'')} d\lambda'' \right].$$

An approximate evaluation of equation (5.24) depends on the value of n. We shall consider two distinct cases (i) n=1, (ii) n > 1.

Case I : n = 1

We shall assume that $\gamma_m(\lambda)$ is a smooth function of λ at $\lambda=\lambda^*$. We may then approximate

$$\int_{\lambda'}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda'')}{\beta(\lambda'')} d\lambda'' \approx \int_{\lambda'}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda'')}{a(\lambda^*-\lambda'')} d\lambda''$$

$$= \frac{2\gamma_m(\lambda^*)}{a} \ln \frac{\lambda^*-\lambda'}{\delta}$$

Thus,

$$\exp \int_{\lambda'}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda'')d\lambda''}{\beta(\lambda'')} = \left[\frac{\lambda^*-\lambda'}{\delta}\right]^{\frac{2\gamma_m(\lambda^*)}{a}} = \left[\frac{\lambda^*-\lambda'}{\delta}\right]^{\alpha}$$

where $\alpha = 2\gamma_m(\lambda^*)/a$.

Assuming further that $Z_{14}^{(1)}$ is a smooth function at $\lambda = \lambda^* \neq 0$

$$\int_{\lambda}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda')\, Z_{14}^{(1)}(\lambda')}{\beta(\lambda')} \exp \int_{\lambda'}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda'')d\lambda''}{\beta(\lambda'')}$$

$$\approx \frac{2\gamma_m(\lambda^*)Z_{14}^{(1)}(\lambda^*)}{a\delta^{\alpha}} \int_{\lambda}^{\lambda^*-\delta} (\lambda^*-\lambda')^{\frac{2\gamma_m(\lambda^*)}{a}-1} d\lambda'$$

$$= Z_{14}^{(1)}(\lambda^*) \frac{(\lambda^*-\lambda)^{\alpha}}{\delta^{\alpha}} - Z_{14}^{(1)}(\lambda^*) \qquad (5.25)$$

This yields

$$K(\lambda) \simeq C_o \frac{(\lambda^*-\lambda)^{-\alpha}}{\delta^{-\alpha}} - Z_{14}^{(1)}(\lambda^*) + Z_{14}^{(1)}(\lambda)\frac{(\lambda^*-\lambda)^{-\alpha}}{\delta^{-\alpha}} \qquad (5.26)$$

This requires that

$$K(\lambda) + Z_{14}^{(1)}(\lambda^*) \simeq (\lambda^*-\lambda)^{-\alpha} \left[C_o\ \delta^{\alpha} + Z_{14}^{(1)}(\lambda^*)\ \delta^{\alpha}\right] \qquad (5.27)$$

The left hand side is independent of δ and so is the factor $(\lambda^*-\lambda)^{-\alpha}$ on right hand side. This requires that the square bracket is also independent of δ (to the leading order). One can therefore express equation (5.27) as

$$K(\lambda) + Z_{14}^{(1)}\ (\lambda^*) \simeq -\ 4C\ (\lambda^*-\lambda)^{-\alpha} \qquad (5.28)$$

Now from equation (5.19), it follows that

$$Z_{14}^{(1)}(\lambda) = Z_{14}^{(1)}(\lambda^*) - \frac{\gamma_{14}(\lambda\ )}{\lambda^*}(\lambda^*-\lambda) + 0\ (\lambda^*-\lambda)^2 \qquad (5.29)$$

Equations (5.20), (5.28), (5.29) then imply that for $n-1 < \alpha < n$ $n = 0,1,\ldots\ldots\ldots$

$$-\frac{1}{8}\frac{\beta(\lambda)}{\gamma_m(\lambda)}\ \frac{d}{d\lambda}K(\lambda) \simeq C(\lambda^*-\lambda)^{-\alpha} + \text{terms of order}$$

$$(\lambda^*-\lambda)^{-\alpha+1}\ \ldots\ldots\ (\lambda^*-\lambda)^{-\alpha+n}\ +$$

$$\frac{1}{4}\,\frac{\gamma_{14}(\lambda^*)}{\lambda^*}\,(\lambda^*-\lambda) + \ldots\ldots.$$

and for $-2 < \alpha < 0$

$$-\frac{1}{8}\,\frac{\beta(\lambda)}{\gamma_m(\lambda)}\,\frac{dK(\lambda)}{d\lambda} = C(\lambda^*-\lambda)^{-\alpha} + \frac{1}{4}\,\frac{\gamma_{14}(\lambda^*)}{\lambda^*}\,(\lambda^*-\lambda) + \text{higher order terms}$$

while for $\alpha \le -2$

$$-\frac{1\beta(\lambda)}{8\gamma_m(\lambda)}\,\frac{dK(\lambda)}{d\lambda} = \frac{1}{4}\frac{\gamma_{14}(\lambda^*)}{\lambda^*}\,(\lambda^*-\lambda) + \text{terms of order}$$

$$(\lambda^*-\lambda)^2 + \ldots\ldots.\quad (\lambda^*-\lambda)^{-[\alpha]} + C\,(\lambda^*-\lambda)^{-\alpha}\ \ldots. \qquad (5.30)$$

[The terms in equation (5.30), not explicitly calculated, are not needed in the future discussions].

The above expression shows that if $\alpha = \frac{2\gamma_m(\lambda^*)}{a}$ is positive, i.e. if $\gamma_m(\lambda^*) > 0$, since as is pointed out in the beginning, $a > 0$ for the case we have in hand, the coefficient of $\langle O_4\rangle^R$ in fact blows up as $\lambda \to \lambda^*$. Thus, the trace anomaly instead of vanishing at a fixed point, in fact, blows up near $\lambda=\lambda^*$, (assuming that $\langle O_4\rangle^R$ is smooth near $\lambda=\lambda^*$. In this connection see section 5.4). Moreover, the dominant behavior of the trace anomaly is, in the case $\alpha > -1$, solely determined by $\alpha = \frac{2\gamma_m(\lambda^*)}{a}$ and does not depend on, say, extra anomalous dimension.

Case II : $n > 1$

This case can be dealt with in a similar manner. We sketch the derivation, stating the approximations made

$$\int_{\lambda'}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda'')}{\beta(\lambda'')}\, d\lambda'' \approx \int_{\lambda'}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda^*)\,d\lambda''}{a(\lambda^*-\lambda'')^n}$$

$$= \frac{}{-n+1}\left[\frac{1}{(\lambda^*-\lambda')^{n-1}} - \frac{1}{(\delta)^{n-1}}\right] \tag{5.31}$$

$$\int_{\lambda}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda')Z_{14}^{(1)}(\lambda')}{\beta(\lambda')} \exp\left[\int_{\lambda'}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda'')\,d\lambda''}{\beta(\lambda'')}\right] d\lambda'$$

$$= \int_{\lambda}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda')Z_{14}^{(1)}(\lambda')}{\beta(\lambda')} \exp -\left[\frac{\alpha}{(n-1)}\frac{1}{(\lambda^*-\lambda')^{n-1}}\right] \exp\left[\frac{\alpha}{(n-1)\delta^{n-1}}\right] d\lambda'$$

$$\approx \int_{\lambda}^{\lambda^*-\delta} \frac{2\gamma_m(\lambda^*)\ Z_{14}^{(1)}(\lambda^*)}{a\ (\lambda^*-\lambda')^n} \exp\left[\frac{\alpha}{(n-1)\delta^{n-1}}\right] \exp\left[\frac{-\alpha}{(n-1)(\lambda^*-\lambda')^{n-1}}\right] d\lambda'$$

A change of variables $-\dfrac{1}{(n-1)(\lambda^*-\lambda')^{n-1}} = \xi$ gives

$$= -\alpha\, Z_{14}^{(1)}(\lambda^*) \exp\left[\frac{\alpha}{(n-1)\delta^{n-1}}\right] \int_{\xi_1}^{\xi_2} e^{\alpha\xi}\, d\xi$$

$$= -Z_{14}^{(1)}(\lambda^*)\exp\left[\frac{\alpha}{(n-1)\delta^{n-1}}\right]\left[-\exp\left[\frac{-\alpha}{(n-1)(\lambda^*-\lambda)^{n-1}}\right]+\exp\left[\frac{-\alpha}{(n-1)\delta^{n-1}}\right]\right]$$

and thus

$$K(\lambda) + Z_{14}^{(1)}(\lambda^*) \simeq Z_{14}^{(1)}(\lambda^*)\exp\left[\frac{\alpha}{(n-1)(\lambda^*-\lambda)^{n-1}}\right]\exp\left[\frac{-\alpha}{(n-1)\delta^{n-1}}\right] +$$

$$C_o \exp\left[\frac{-\alpha}{(n-1)\delta^{n-1}}\right] \exp\left[\frac{\alpha}{(n-1)(\lambda^*-\lambda)^{n-1}}\right]$$

$$= \exp\left[\frac{\alpha}{(n-1)(\lambda^*-\lambda)^{n-1}}\right] \left[Z_{14}^{(1)}(\lambda^*) \exp\left[\frac{-\alpha}{(n-1)\delta^{n-1}}\right]\right.$$

$$\left. + \ C_o \exp\left(\frac{-\alpha}{(n-1)\delta^{n-1}}\right)\right] \qquad (5.32)$$

As in the earlier case, the square bracket must be independent of δ (to the leading order) and thus

$$K(\lambda) + Z_{14}^{(1)}(\lambda^*) \simeq C' \exp\left[\frac{\alpha}{(n-1)(\lambda^*-\lambda)^{n-1}}\right], \ \alpha > 0 \qquad (5.33a)$$

$$\simeq \frac{1}{4} \frac{\gamma_{14}(\lambda^*)}{\lambda^*} (\lambda^*-\lambda) + \ldots\ldots, \quad \alpha < 0 \qquad (5.33b)$$

Again if $\alpha = \dfrac{2\gamma_m(\lambda^*)}{a}$ is positive (i.e. if $\gamma_m(\lambda^*) > 0$), the coefficient of $\langle 0_4\rangle^R$ in the expression for $\theta_\mu^{\mu\,imp}$ diverges exxponentially as $\lambda \to \lambda^*$ (from below). Thus, in this case also the trace anomaly blows up as $\lambda \to \lambda^*$. The behavior of trace anomaly is again solely determined by $\gamma_m(\lambda^*)$. On the other hand if $\alpha < 0$, equation (5.33b) yields the correct behavior of the anomaly coefficient near $\lambda = \lambda^*$.

5.4 Scaling Equation

Consider, for simplicity, a multiplicatively renormalizable operator 0, having anomalous dimension γ_o. The scaling equation derived from RG equation and dimensional analysis read [20].

$$\left[\sum_\nu \sum_{i=1}^{n} p_{i\nu} \frac{\partial}{\partial p_{i\nu}} - \beta(\lambda) \frac{\partial}{\partial\lambda} + m(1-\gamma_m) \frac{\partial}{\partial m} + n(1+\gamma) - 4 + \gamma_o\right] \times$$

$$\times\, \Gamma_o^{(n)}(p_1, p_2, \ldots p_n, \lambda, m, \mu) = 0 \qquad (5.34)$$

It is usually assumed that at the fixed point $\beta(\lambda^*) = 0$ and hence the second term can be dropped when λ is in the neighbourhood of λ^*. This leads, in the massless limit, to

$$\left[\sum_{i=1}^{n} \sum_{\nu} p_{i\nu} \frac{\partial}{\partial p_{i\nu}} + n(1+\gamma) - 4 + \gamma_o \right]$$

$$\times \Gamma_o^{(n)} (p_1 \ldots\ldots p_n, \lambda^*, 0, \mu) = 0 \qquad (5.35)$$

The above equation would imply that in the large momentum limit, $\Gamma_o^{(n)}$ scales by the scaling dimension $-n(1+\gamma) + 4-\gamma_o$ at the fixed point.

In this section, we wish to present an example of a case where the $\beta(\lambda) \frac{\partial}{\partial\lambda}$term cannot be dropped and instead leads to a nontrivial contribution and a different scaling dimension as compared to that in Eq. (5.35). Consider $0 = \theta_{\mu}^{\mu\,imp}$ is a finite operator and hence $\gamma_o = 0$. For simplicity, consider the massless case. Then $\langle 0_3 \rangle^R = 0$. Further, for concreteness, consider the possibility n=1 in Eq. (5.23) and let $\gamma_m(\lambda^*) > 0$. Then the leading behavior of Γ_o near $\lambda = \lambda^*$ is given solely by the term proportional to $\langle 0_4 \rangle^R$ (provided it does not vanish)

$$\Gamma_o^{(n)} \approx -\frac{1}{8} \frac{\beta(\lambda)}{\gamma_m(\lambda)} \frac{d}{d\lambda} K(\lambda)\, \Gamma_{o_4}^{(n)}$$

$$\approx C(\lambda^*-\lambda)^{\alpha}\, \Gamma_{o_4}^{(n)} (p_1, p_2, \ldots\ldots p_n, \lambda^*, \mu)$$

$$\alpha > 0 \qquad (5.36)$$

assuming that Γ_{o_1} are smooth functions of λ near $\lambda=\lambda^*$ [A comment about this is made later]. Then one has, from Eq. (5.34),

$$\left[\sum_{i=1}^{n} \sum_{\nu} p_{i\nu} \frac{\partial}{\partial p_{i\nu}} - \beta(\lambda) \frac{\partial}{\partial\lambda} + n(1+\gamma) -4\right] \Gamma_o^{(n)}(p_1 \ldots p_n, \lambda, \mu) = 0 \tag{5.37}$$

Now the term

$$\begin{aligned}
- \beta(\lambda) \frac{\partial}{\partial\lambda} \Gamma_o^{(n)} &\approx - \beta(\lambda) \frac{\partial}{\partial\lambda} C(\lambda^*-\lambda)^{-\alpha} \Gamma_{o_4}^{(n)} \\
&\approx - a(\lambda^*-\lambda) \, C \, \alpha(\lambda^*-\lambda)^{-\alpha-1} \Gamma_{o_4}^{(n)} \\
&= - 2\gamma_m(\lambda^*) \, C(\lambda^*-\lambda)^{-\alpha} \Gamma_{o_4}^{(n)} \\
&\simeq - 2\gamma_m(\lambda^*) \, \Gamma_o^{(n)} \, (p_1, \ldots p_n, \lambda^*, \mu)
\end{aligned} \tag{5.38}$$

contributes nontrivially to the equation, leading to the different scaling dimension $2\gamma_m(\lambda^*) - n(1+\gamma) + 4$.

In the above discussion, we have seen that $\Gamma_o^{(n)}$ has a nontrivial behavior near the fixed point if $\alpha > 0$ (i.e. if $\gamma_m(\lambda^*) > 0$). This observation, based on Eq.(5.36), relies on the fact $\Gamma_o^{(n)}$ is a smooth function near $\lambda = \lambda^*$. If $\Gamma_{o_4}^{(n)}$ had a nontrivial behavior near $\lambda = \lambda^*$ such as, for example, to cancel the singular behavior of the anomaly coefficient $-\frac{1}{8} \frac{\beta(\lambda)\, d}{\gamma_m(\lambda) d\lambda} K(\lambda)$ our conclusion would not be valid. It is normally assumed that for any operator O, $\Gamma_o^{(n)}$ has a smooth behavior near a fixed point.

But this cannot be expected of $\theta_{\mu}^{\mu imp}$ because it contains $\frac{g(\varepsilon)}{1-n}\partial^2\phi^2$ term which has been constructed in a nontrivial manner in perturbation series. $\langle\theta_{\mu}^{\mu imp}\rangle$ explicitly contains $K(\lambda)$ which has been chosen to satisfy a differential condition of Eq. (5.20). This is the justification for allowing $\Gamma_o^{(n)}$ to possess a nontrivial behavior near $\lambda=\lambda^*$, while $\Gamma_{oi}^{(n)}$ $(i=1,2,3,4)$ have been assumed to be smooth and nonvanishing near $\lambda=\lambda^*$.

A similar conclusion holds in the cases $n > 1$ and $\alpha > 0$ as is verified easily.

5.5 Alternate derivation of the scaling dimension of $\Gamma_o^{(n)}$

In the preceeding section we showed that the scaling dimension for $\Gamma_o^{(n)}$ differs from that naively expected. In this section, we shall present an alternative but extremely straightforward derivation of these results.

From Eq. (5.14) we have for $m_o = 0$

$$
\begin{aligned}
\Gamma_o^{(n)}(e^t p_i,\lambda,\mu) = {} & \frac{\beta(\lambda)}{4\lambda}\,\Gamma_{o_1}^{(n)}(e^t p_i,\lambda,\mu) \\
& - \left(1+\gamma - \frac{\beta(\lambda)}{4\lambda}\right)\Gamma_{o_2}^{(n}(e^t p_i,\lambda,\mu) \\
& - \frac{1}{8}\,\frac{\beta(\lambda)}{\gamma_m(\lambda)}\,\frac{d}{d\lambda}K(\lambda)\,\Gamma_{o_4}^{(n)}(e^t p_i,\lambda,\mu)
\end{aligned}
\tag{5.39}
$$

We want to see the behavior of $\Gamma^{(n)}(p_i,\lambda,\mu)$ under small rescaling of p_i when $\lambda \approx \lambda^*$.

If $\gamma_m(\lambda^*) > 0$, coefficient of $\Gamma_o^{(n)}$ in Eq. (5.39) is relatively small and that of $\Gamma_{o_2}^{(n)}$ is approximately $-[1+\gamma(\lambda^*)]$, a finite number while that of $\Gamma_{o_4}^{(n)}$ is large, near the fixed point. Hence, provided that $\Gamma_{o_4}^{(n)}$ is not zero or negligible,

$$\Gamma_o^{(n)}(e^t p_{i,} \lambda, \mu) \approx -\frac{1}{8}\frac{\beta(\lambda)}{\gamma_m(\lambda)}\frac{d}{d\lambda}K(\lambda)\,\Gamma_{o_4}^{(n)}(e^t p_{i,}\lambda, \mu) \tag{5.40}$$

Now $O_4 = \partial^2\phi^2$ is a multiplicatively renormalizable operator with anomalous dimension $\gamma_{o_4} = 2\gamma_m$. Hence,

$$\Gamma_o^{(n)}(e^t p_{i,}\lambda,\mu) \simeq \exp\left\{\int_o^t dt'\left\{\left[2\gamma_m(\lambda(t'))\right] - n\gamma(\lambda(t'))\right\} + (4-n)t\right\} \times$$

$$\times\,\Gamma_{o_4}^{(n)}(p_{i,}\lambda^*,\mu)$$

For $\lambda \simeq \lambda^*$ and t small this becomes,

$$\simeq \exp\left\{4+2\gamma_m(\lambda^*) - n[1+\gamma(\lambda^*)]\right\} t \times$$

$$\times\,\Gamma_{o_4}^{(n)}(p_{i,}\lambda^*, \mu) \tag{5.41}$$

This implies for

$$\Gamma_o^{(n)}(e^t p_{i,}\lambda, \mu) \approx \exp\left\{4+2\gamma_m(\lambda^*) - n\,[1+\gamma(\lambda^*)]\right\} t \times$$

$$\times\,\Gamma_o^{(n)}(p_{i,}\lambda^*, \mu) \tag{5.42}$$

giving rise to the same anomalous dimension as in Sec. 5.4, it is however important to note that in Sec. (5.4) the term $2\gamma_m(\lambda^*)$

arose out of the coefficient of Γ_{O_4} through the term $\beta(\lambda)\,\frac{\partial}{\partial\lambda}\,\Gamma_o$; while in Eq. (5.42), it arises as the anomalous dimension of O_4.

5.6 Behavior of $\langle\theta_{\mu\nu}^{imp}\rangle$ near the fixed point

Uptil now we have discussed the behavior of $\langle\theta_{\mu}^{\mu\,imp}\rangle$ near the fixed point λ^* and derived the scaling properties of $\Gamma_{\theta_{\mu}^{\mu}imp}$. In this section our aim is to derive the expression for $\langle\theta_{\mu\nu}^{imp}\rangle$ and also obtain the leading behavior of Green's functions with one insertion of $\theta_{\mu\nu}^{imp}$ as all the external momenta are scaled by a common factor. In order to do this, first we shall have to study the renormalization of $\theta_{\mu\nu}^{imp}$.

Renormalization of $\theta_{\mu\nu}^{c}$, the canonical energy momentum tensor has been discussed thoroughly by Brown [11]. We make use of his results to obtain an expression for $\theta_{\mu\nu}^{imp}$ in terms of renormalized operators. In order to do this, it is necessary to split $\theta_{\mu\nu}^{imp}$ in terms of the traceless part and the trace

$$\theta_{\mu\nu}^{imp} \equiv \theta_{\mu\nu}^{TL\;imp} + \frac{\eta_{\mu\nu}}{n}\,\theta_{\sigma}^{\sigma\;imp} \tag{5.43}$$

with

$$\theta_{\mu\nu}^{TL\;imp} = \partial_\mu\phi\,\partial_\nu\phi - \frac{1}{n}\,\eta_{\mu\nu}\,\partial_\sigma\phi\,\partial^\sigma\phi + \left[\frac{n-2}{4(1-n)} - \frac{\epsilon\,\tilde{g}\,(\epsilon)}{4(1-n)}\right] \times$$
$$(\partial_\mu\partial_\nu - \frac{1}{n}\,\eta_{\mu\nu}\partial^2)\phi^2 \tag{5.44}$$

and

$$\theta_{\mu}^{\mu\,imp} = -\frac{\epsilon}{4}\,O_1 - \frac{n}{2}\,O_2 + O_3 + \left[g(\epsilon) + \frac{1}{2} - \frac{n}{4}\right]O_4 \tag{5.45}$$

where

$$O_1 = \phi\,(\partial^2 + m_o{}^2)\,\phi \qquad O_2 = \frac{\delta S}{\delta \phi}\,\phi$$

$$O_3 = m_o^2\,\phi^2 \qquad O_4 = \partial^2\,\phi^2 \tag{5.46}$$

To make contact with Brown's results, we note that

$$O_1 = O_2 - 4\left[\frac{\lambda_o}{4!}\,\phi^4\right] \tag{5.47}$$

We also let

$$O_5 = \phi\, t_{\mu\nu}\,\phi \equiv \phi\left[n\,\partial_\mu\,\partial_\nu - \eta_{\mu\nu}\,\partial^2\right]\phi$$

$$O_6 = t_{\mu\nu}\,\phi^2 \tag{5.48}$$

$$O_7 \equiv \phi^2 \tag{5.49}$$

Then one has,

$$\theta_{\mu\nu}^{TL\,imp} = -\frac{1}{n}\,\phi t_{\mu\nu}\,\phi + \left[\frac{1}{2} + \frac{n-2}{4(1-n)} - \frac{\varepsilon\,\tilde{g}\,(\varepsilon)}{4(1-n)}\right]\frac{1}{n}\,t_{\mu\nu}\,\phi^2 \tag{5.50}$$

Using [11]

$$\langle\phi\,t_{\mu\nu}\phi\rangle^{UR} = \langle\phi\,t_{\mu\nu}\phi\rangle^{R} + \frac{1}{n-1}\left\{-\left[Z_m^{-1}(n)\,-1\right] - \frac{1}{4}\left[Z_m^{-1}(1)\,-1\right]\right.$$
$$\left. - \frac{\epsilon\,Z_{14}(n)}{4} + \frac{3}{4}\,Z_{14}(1)\right\} t_{\mu\nu}\,\langle\phi^2\rangle^R \tag{5.51}$$

Here Z[1] means a renormalization constant Z evaluated at n=1 i.e. at $\epsilon=3$ and $\langle\ \rangle^R$ denotes Green's functions of a renormalized

operator. Z_{14} is defined via

$$< O_1 >^{UR} = \sum_{i=1}^{4} Z_{1i} < O_i >^R \qquad (5.52)$$

Using Eq. (5.51) in Eq. (5.50), collecting terms and using

$\langle\phi^2\rangle^{UR} = Z_m^{-1} \langle\phi^2\rangle^R$; one obtains

$$\langle\theta_{\mu\nu}^{TL}{}^{imp}\rangle = -\frac{1}{n}\langle\phi\ t_{\mu\nu}\phi\rangle^R + \Bigg\{\left[\frac{1}{2} + \frac{n-2}{4(1-n)} - \frac{\epsilon\ \tilde{g}(\epsilon)}{4(1-n)}\right]Z_m^{-1}$$

$$- \frac{1}{n-1}\left[\frac{n}{4}\ (Z_m^{-1}(n) - 1) - \frac{1}{4}\ (Z_m^{-1}(1) - 1)\right]$$

$$+ \frac{1}{n-1}\left[\frac{\epsilon}{4}\ Z_{14}(n) - \frac{3}{4}\ Z_{14}(1)\right]\Bigg\}\ \frac{t_{\mu\nu}}{n}\ \langle\phi^2\rangle^R \qquad (5.53)$$

$$= -\frac{1}{n}\langle\phi\ t_{\mu\nu}\ \phi\rangle^R + \Bigg\{\frac{1}{4} + \frac{Z_m^{-1}(1)}{4(n-1)} - \frac{3\ Z_{14}(1)}{4\ (n-1)}$$

$$- \frac{\epsilon\ \tilde{g}\ (\epsilon)\ Z_m^{-1} + \epsilon\ Z_{14}(n)}{4(1-n)}\Bigg\}\ \frac{t_{\mu\nu}}{n}\ \langle\phi^2\rangle \qquad (5.54)$$

As discussed in the section 5.2, finiteness of $\theta_{\mu\nu}^{imp}$ requires that $\epsilon\ [Z_{14} + \tilde{g}\ (\epsilon)\ Z_m^{-1}]$ must be finite. Hence in the notations used there

$$\left[\epsilon\ Z_{14}(n) + \epsilon\ \tilde{g}\ (\epsilon)\ Z_m^{-1}\right]_{n=4} = Z_{14}^{(1)} + K(\lambda) \qquad (5.55)$$

and thus one has

$$\langle\theta_{\mu\nu}^{TL}{}^{imp}\rangle = \frac{1}{n}\langle\phi\ t_{\mu\nu}\phi\rangle^R + \Bigg\{\frac{1}{4} + \frac{Z_m(1)}{4(n-1)} - \frac{3\ Z_{14}(1)}{4\ (n-1)}\Bigg\}\ \frac{t_{\mu\nu}}{n}\ \langle\phi^2\rangle^R$$

$$- \frac{1}{4} \left[K(\lambda) + Z_{14}^{(1)} (\lambda)\right] \frac{t_{\mu\nu}}{n(1-n)} \langle\phi^2\rangle^R \qquad (5.56)$$

$$+ \text{ term of } O(\epsilon)$$

On the other hand from [Section 5.2]

$$\frac{\eta_{\mu\nu}}{n} \langle\theta^\sigma_\sigma\rangle = \frac{\eta_{\mu\nu}}{n} \left\{\frac{\beta(\lambda)}{4\lambda} \langle O_1\rangle^R + \left[\frac{\beta(\lambda)}{4\lambda} -\gamma-1\right] \langle O_2\rangle^R - (\gamma_m -1) \langle O_3\rangle^R \right.$$

$$\left. - \left[Z_{14}^{(1)} (\lambda) + K(\lambda)\right] \partial^2 \langle\phi^2\rangle^R \right\}$$

$$+ O(\epsilon) \qquad (5.57)$$

Thus Eq. (5.43) yields

$$\langle\theta^{imp}_{\mu\nu}\rangle = - \frac{1}{n} \langle\phi \; t_{\mu\nu}\phi\rangle^R + \left\{\frac{1}{4} + \frac{Z_m^{-1}(n)}{4(n-1)} - \frac{3 \; Z_{14}(1)}{4 \; (n-1)} \right\} \frac{t_{\mu\nu}}{n} \langle\phi^2\rangle^R$$

$$+ \frac{\eta_{\mu\nu}}{n} \left\{\frac{\beta(\lambda)}{4\lambda} \langle O_1\rangle^R + \left(\frac{\beta(\lambda)}{4\lambda} - \gamma\right) \langle O_2\rangle^R - (\gamma_m) \langle O_3\rangle^R \right\}$$

$$- \frac{1}{4} \left[Z_{14}^{(1)} (\lambda) + K(\lambda)\right] \frac{(\partial_\mu\partial_\nu - \eta_{\mu\nu} \; \partial^2)}{1-n} \langle\phi^2\rangle^R$$

$$+ O(\epsilon) \qquad (5.58)$$

After having obtained the express, ion for $\langle\theta^{imp}_{\mu\nu}\rangle$, we go further to study the high energy behavior of Green's functions with one insertion of $\theta^{imp}_{\mu\nu}$.

We shall assume the existence of an ultraviolet attractive fixed point $\lambda=\lambda^*$ and that $\gamma_m(\lambda^*) > 0$. We shall denote by $G^{(n)t}_{\mu\nu}$,

the n point truncated Green's function of $\theta^{imp}_{\mu\nu}$. Similarily we shall denote by $G^{(n)t}_{i(i=1....7)}$, the n point truncated renormalized Green's function of O_i (i=1...7), defined in eqs. (5.46), (5.48) and (5.49). Then from Eq. (5.58) we have

$$
\begin{aligned}
G^{(n)t}_{\mu\nu}\left(e^t p_i, \lambda, m, \mu\right) &= \frac{-1}{4}\, G^{(n)t}_5\left(e^t p_i, \lambda, m, \mu\right) \\
&+ \frac{1}{4}\left\{\frac{1}{4} + \frac{Z_m^{-1}(1)}{12} - \frac{Z_{14}(1)}{4}\right\} G^{(n)t}_5\left(e^t p_i, \lambda, m, \mu\right) \\
&+ \frac{\eta_{\mu\nu}}{4}\left\{\frac{\beta(\lambda)}{4\lambda}\, G^{(n)t}_1\left(e^t p_i, \lambda, m, \mu\right)\right. \\
&\qquad + \left[\frac{\beta(\lambda)}{4\lambda} - \gamma(\lambda) - 1\right] G^{(n)t}_2\left(e^t p_i, \lambda, m, \mu\right) \\
&\qquad \left. - (\gamma_m(\lambda) - 1)\, G^{(n)t}_2\left(e^t p_i, \lambda, m, \mu\right)\right\} \\
&+ \frac{\beta(\lambda)}{24\gamma_m(\lambda)}\, \frac{dK(\lambda)}{d\lambda}\, e^{2t}\left(q_\mu q_\nu - q^2 \eta_{\mu\nu}\right) G^{(n)t}_7\left(e^t p_i, \lambda, m, \mu\right)
\end{aligned}
\tag{5.59}
$$

We shall assume that $G^{(n)t}_i$ (i=1,2....7), $Z_m^{-1}(1)$, $Z_{14}(1)$, $\gamma(\lambda)$ and $\gamma_m(\lambda)$ are smooth functions of λ near $\lambda=\lambda^*$. However, as shown in section 5.3, the coefficient of the last term $\frac{\beta(\lambda)}{24\gamma_m(\lambda)}\frac{dK(\lambda)}{d\lambda}$ can blow up near the fixed point λ^*. If this happens the last term dominates the rest of the terms near a fixed point if $q \neq 0$, (q being the momentum entering via $\theta^{imp}_{\mu\nu}(x)$).

Thus we have,

$$G^{(n)t}_{\mu\nu}(e^t p_{i,} \lambda, m, \mu) \simeq \frac{\beta(\lambda)}{24\gamma_m(\lambda)} \frac{dK(\lambda)}{d\lambda} e^{2t}(q_\mu q_\nu - \eta_{\mu\nu} q^2) G^{(n)t}_7(e^t p_{i,} \lambda, m, \mu)$$

(5.60)

Now, $O_7 = \phi^2$ is a multiplicatively renormalizable operator with anomalous dimension $2\gamma_m$. The renormalization group properties of the truncated Green's functions with external lines renormalized on-shell are the same as those of the corresponding proper vertices with the anomalous dimension term deleted.

Hence we have,

$$G^{(n)t}_7(e^t p_{i,} \lambda, m, \mu) \simeq e^{\int_0^t dt' \{2\gamma_m[\lambda(t')]\} + (2-n)t}$$

$$\times\, G^{(n)t}_7 \,(p_{i,}\ \lambda^*,\ \bar{m}\,(t) e^{-t},\ \mu) \qquad (5.61)$$

with $\bar{m}(t)$ defined by [20]

$$\frac{\partial\, \bar{m}(t)}{\partial t} = \bar{m}(t)\, \gamma_m\, [\lambda(t)] \text{ with } \bar{m}(0) = m$$

and thus

$$G^{(n)t}_{\mu\nu}(e^t p_{i,} \lambda, m, \mu) \simeq \frac{\beta(\lambda)}{24\gamma_m(\lambda)} \frac{dK(\lambda)}{d\lambda} (q_\mu q_\nu - q^2 \eta_{\mu\nu})$$

$$\times\, e^{\int_0^t dt' \{2\gamma_m[\lambda(t')] + 4 - n\}}$$

$$\times\, G^{(n)t}_7 \,(p_{i,}\ \lambda^*,\ \bar{m}\,(t) e^{-t},\ \mu)$$

$$\simeq e^{\int_0^t dt' \{2\gamma_m\,[\lambda(t')] + 4 - n\}}$$

$$\times\, G^{(n)t}_{\mu\nu} \,(p_{i,}\ \lambda^*,\ \bar{m}(t)\ e^{-t},\ \mu) \qquad (5.62)$$

We note that in deriving the above relation, we have defined the symbol $G^{(n)t}_{\mu\nu}(p_i, \lambda^*, \bar{m}(t)e^{-t}, \mu)$, since $G^{(n)t}_{\mu\nu}$ receives the main contribution from $G^{(n)t}_7$ term near the fixed point. $G^{(n)t}_7$ is a smooth function near $\lambda=\lambda^*$ but its coefficient is not.

As $\lambda(\mu) \simeq \lambda^*$, in the leading approximation $\gamma_m(\lambda) \simeq \gamma_m(\lambda^*)$ (assuming $\gamma_m(\lambda^*)$ is non zero). One then has

$$G^{(n)t}_{\mu\nu}(e^t p_i, \lambda, m, \mu) \simeq e^{[4-n+2\gamma_m(\lambda^*)]t} G^{(n)t}_{\mu\nu}(p_i, \lambda^*, \mu, \bar{m}(t)e^{-t}) \tag{5.63}$$

Thus $\theta^{imp}_{\mu\nu}$ behaves as if it has an anomalous dimension $2\gamma_m(\lambda)$. This leads to a nontrivial high energy behaviour provided $\gamma_m(\lambda^*) > 0$. It is a peculiarity of this energy momentum tensor.

5.7 Behavior of $\langle\theta^{imp}_{\mu\nu}\rangle$ near the fixed point in scalar QED and Weinberg-Salam model

In chapter 3 and 4, we investigated the case of obtaining a unique, finite energy momentum tensor with the help of a less restrictive 'RG covariance' criterion, in scalar QED and W-S model. We showed that in such a case, $\bar{k}$, which appears in the coefficient of the improvement term

$$H_o = (G + \bar{k}\, Z_m)(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\,\phi^\dagger\phi$$

is fixed uniquely and can be expanded perturbatively in the coupling constants. Here we shall study the high energy behavior

of Green's functions with the insertion of energy momentum tensor containing $\overline{k}$, which has perturbative expansion in coupling constants, in scalar QED and W-S model.

Case I : Scalar QED at $\lambda = 0$.

We shall assume the existence of an ultraviolet stable fixed point e^* in the theory. From our analysis in chapter 3, we know that the equation to be satisfied by $\overline{k}(e^2)$ is

$$2e\,\overline{\beta}^e \frac{d}{de^2}\,\overline{k}(e^2) - 2\gamma_m(e^2)\overline{k}(e^2) = (\frac{e}{2}\frac{d}{de})G^{(1)}(e^2) \qquad (5.64)$$

The equation written above has an exact solution of the kind we had obtained in the case of $\lambda\phi^4$ theory given by equation (5.24) in section 5.3.

The constant of integration is fixed by imposing the boundary condition that $\overline{k}(e^2)$ is a perturbative series in e^2. We shall assume that the fixed point e^* is approached from below and β function is analytic close to the fixed point and it is given by

$$e\overline{\beta}^e(e) \simeq b(e^{*2} - e^2) \qquad \text{where } b > 0.$$

Also we assume that

$$\gamma_m(e^2) \simeq \gamma_m(e^{*2})$$

and $$\frac{e}{2}\frac{d}{de}\,G^{(1)}\,(e^2) = \xi(e^2) \simeq \xi(e^{*2})$$

With these approximations, equation 5.64 has the solution of the form

$$\bar{k}(e^2) \simeq c\,(e^{*2} - e^2)^{-\alpha} \quad \text{where } \alpha = 2\gamma_m(e^{*2})/b$$

Hence $\bar{k}(e^2)$ can diverge near the UV stable fixed point e^* if $\gamma_m(e^{*2}) > 0$. A similar analysis holds if $e\bar{\beta}^e(e) \simeq b(e^{*2} - e^2)^n$ $(n > 1)$

Thus the Green's functions with the insertion of $\theta_{\mu\nu}^{imp}$ containing $\bar{k}(e^2)$, can show non trivial behavior near the fixed point e^* if $\gamma_m(e^{*2}) > 0$.

Case II : The behavior of $\bar{k}(\lambda, e^2)$ in scalar QED near the non trivial UV stable fixed point λ^*, $e^* \neq 0$, which we assume to exist in the theory, can be determined in a self consistent way. We make an assumption that $\bar{k}(\lambda, e^2)(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\langle\phi^\dagger\phi\rangle^R$ is the dominant term as compared to the other terms in the renormalized energy momentum tensor.

The scaling law for the n point proper vertices with the insertion of the operator $\theta_{\mu\nu}^{imp}$ is given by

$$\Gamma^{(n)}_{\theta_{\mu\nu}}(e^t p_i, \lambda, e, h, m, \mu) = e^{\int_0^t [4-n-n\gamma(\lambda(t'), e(t')]dt'}\,\Gamma^{(n)}_{\theta_{\mu\nu}}(p_i, \bar{\lambda}(t), \bar{e}(t), \bar{h}(t), \bar{m}(t)e^{-t}, \mu)$$

Making use of the assumption that we have made, we can write the following expression:

$$k(\lambda,e)\ \Gamma^{(n)}_{\phi^*\phi}\ (e^t p_i, \lambda, e, m, \mu)$$

$$= e^{\int_0^t [\,4-n-n\gamma(\lambda(t'),e(t'))]\,dt'}\ k(\bar{\lambda}(t),\bar{e}(t))\Gamma^{(n)}_{\phi^*\phi}\ (p_i, \bar{\lambda}(t),\bar{e}(t),\bar{m}(t)e^{-t},\mu)$$

(5.66)

$\phi^*\phi$ is a multiplicatively renormalizable operator with the anomalous dimension $2\gamma_m$. The n point proper vertices with the insertion of $\phi^*\phi$ operator obey the scaling law:

$$\Gamma^{(n)}_{\phi^*\phi}\ (e^t p_i, \lambda, e, m, \mu) = e^{\int_0^t [4-n-n\gamma(\lambda(t'),e(t'))\ +\ 2\gamma_m(\lambda(t'),e(t'))]\,dt'}\ \Gamma^{(n)}_{\phi^*\phi}(p_i, \bar{\lambda}(t),\bar{e}(t),\bar{m}(t)e^{-t},\mu)$$

(5.67)

Equations (5.66) and (5.67) then imply that

$k(\bar{\lambda}(t),\ \bar{e}(t))$ scales as $e^{\int_0^t 2\gamma_m(\lambda(t'),e(t'))dt'}\ k(\lambda,e)$

For small scaling of momenta near the fixed point, we obtain the behavior of $k(\bar{\lambda}(t),\ \bar{e}(t))$. it goes like $e^{2\gamma_m(\lambda^*,e^*)t}\ k(\lambda^*,e)$.

The same analysis as outlined above can be carried out for W-S model also to determine the behavior of $\bar{k}(\lambda,g^2,g'^2)$ near the non trivial UV stable fixed point λ^*, g^{*2}, $g'^{*2} \neq 0$, which we assume to exist in the theory. We assume that $\bar{k}(\lambda,g^2,g'^2)\ (\partial_\mu\partial_\nu - \eta_{\mu\nu},\partial^2) < \phi^\dagger\phi >^R$ is the most dominating term in the renormalized

energy momentum tensor. Then it follows as in scalar QED that

$k(\bar{\lambda}(t), \bar{g}(t), \bar{g}'(t))$ scales as $e^{2\gamma_m(\lambda^*, g^{*2}, g'^{*2})t}\, k(\lambda, g^2, g'^2)$ as $t \to \infty$.

Hence the Green's functions with the insertion of energy momentum tensor containing $\bar{k}(\lambda, g^2, g'^2)$ can show a non trivial high energy behavior near the fixed point, under consideration, if $\gamma_m(\lambda^*, g^{*2}, g'^{*2}) > 0$.

5.8 $\gamma_m(\lambda)$ in large N-limit

Certain interesting observations made in Sec. 5.3, 5.4, 5.5 and 5.6 in the context of $\lambda\phi^4$ theory, were dependent on the sign of $\gamma_m(\lambda^*)$. As the nonzero fixed points of $\lambda\phi$ theory are not known, and knowledge of $\gamma_m(\lambda^*)$ would require an exact calculation, no definite conclusions about sign of $\gamma_m(\lambda^*)$ can be stated. However, it is possible to obtain $\gamma_m(\lambda^*)$ in a related O(N) invariant ϕ^4 theory, at least, in the large N limit (keeping λN fixed) [26]. In this case, it turns out, as outlined below, that γ_m receives contributions only in $O(\lambda)$ and the result is indeed positive for $\lambda^* > 0$ (λ^* must be positive for $H(\lambda^*)$ to have a lower bound).

It is possible to show, by an analysis of graphs that the only graphs that make leading contribution to $\gamma_m(\lambda)$ in each order are of the kind shown in Fig. 1 together with the counterterm graphs that correspond to subtractions of this graph. This set of graphs can be easily summed to give the result. (here,

a is the loop expansion parameter)

$$Z_m = \sum_{n=0}^{\infty} a^n \; z_m^{(n)} \, \lambda^n$$

with $z_m^{(n)}(\epsilon) = A_n/\epsilon^n$

giving only an ϵ^{-n} term contribution to $z_m^{(n)}(\epsilon)$. Thus, the simple pole contribution comes only in the one loop order and hence $\gamma_m(\lambda)$ receives contribution only from one loop approximation. The result is

$$\gamma_m(\lambda) = \frac{1}{16\Pi^2} \frac{N\lambda}{2} + \text{non leading terms.} \tag{5.68}$$

For $\lambda = \lambda^* > 0$, $\gamma_m(\lambda^*) > 0$ in the limit $N\lambda$ fixed and $N \longrightarrow \infty$.

FIGURE CAPTION

Fig. 1: Diagram contributing to Z_m in n loop approximation in the large N limit.

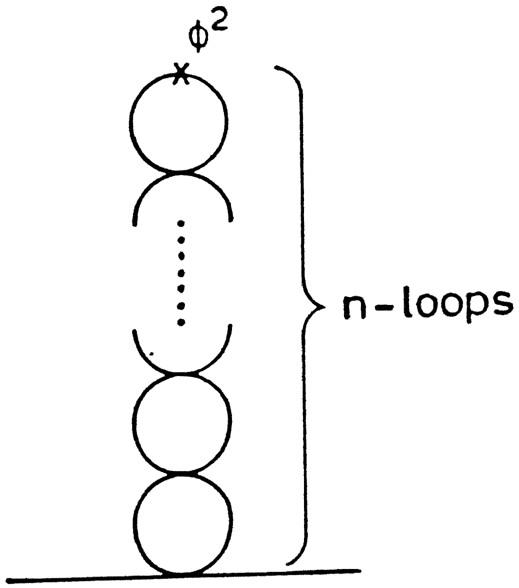

FIG.1

CHAPTER - 6

GRAVITATIONAL SCATTERING IN $\lambda\phi^4$ THEORY

6.1 Introduction:

In section 5.7, we considered the behavior of the energy momentum tensor $\theta_{\mu\nu}^{imp}$ in $\lambda\phi^4$ theory, near the nontrivial ultraviolet fixed point λ^*, where $\theta_{\mu\nu}^{imp}$ is defined below:

$$\theta_{\mu\nu}^{imp} = \partial_\mu\phi\ \partial_\nu\phi - \eta_{\mu\nu}\mathcal{L} + \frac{g(\epsilon)}{(1-n)}(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2 \quad \ldots\ldots\ldots \quad (6.1)$$

This energy momentum tensor, discovered by Collins [8], has finite matrix elements to all orders in perturbation theory. $g(\epsilon)$ is a unique series in non negative powers of ϵ. We have discussed the properties of $\theta_{\mu\nu}^{imp}$ in section 5.1 and hence we shall not repeat them here. In the last chapter, we showed that the Green's functions of the energy momentum tensor given by equation (6.1) can diverge near the fixed point λ^*,under certain conditions specified there. We discussed the renormalization group properties of the Green's functions with the insertion of $\theta_{\mu\nu}^{imp}$. Normally as $\theta_{\mu\nu}^{imp}$ is a finite operator, one would expect its Green's functions to behave as that of an operator of zero anomalous dimension would, leading to a particular definite high energy behavior. However we found that it behaves instead as an operator of anomalous dimension $2\gamma_m(\lambda^*)$ near the λ^*, provided $\gamma_m(\lambda^*) > 0$. This leads to an unexpected high energy behavior of its Green's functions.

Here we are interested in considering some physical processes involving energy momentum tensor $\theta_{\mu\nu}^{imp}$ with its peculiar high energy behavior. The processes we are interested in, can be defined easily since we know that an external gravitational field must couple to the matter via an energy momentum tensor whose matrix elements must be finite. Any renormalizable field theory has an infinite number of energy momentum tensors with finite Green's functions. But the energy momentum tensor of equation (6.1) is unique in the sense that if $\theta_{\mu\nu}^{imp}$ couples to the gravity, the interaction of matter with external gravity is completely determined by flat space parameters as discussed in section 2.1 and also in chapter 5. Thus invoking a principle of `minimal coupling', $\theta_{\mu\nu}^{imp}$ is the most likely candidate for describing the interactions with an external gravity. In our analysis, we shall assume that gravity couples to the matter through this energy momentum tensor. Hence if we consider the processes which are, the scattering of a scalar by a weak external gravitational field and the scattering of two scalars in the presence of weak external gravitational field, the high energy behavior of the differential scattering corsssections for these processes would be nontrivial near $\lambda = \lambda^*$, since the Green's functions with the $\theta_{\mu\nu}^{imp}$ insertion enter the S-matrix calculation. As a side remark, we wish to point out that the field $h_{\mu\nu}$ is taken as the perturbation on the flat space background. In the work presented here, it is shown that for the scattering of a scalar in an external gravity, the crosssection can rise with energy if the anomalous dimension associated with mass, $\gamma_m(\lambda^*)$ is large enough. In the context of

the scattering of two scalars, the differential scatteringcross-section has a piece, that can, if $\gamma_m(\lambda^*) > 0$, rise with energy.

We know that $\lambda\phi^4$ theory is not physically relevant and the results that we have stated above are far from being tested phenomenologically. To that end, we wish to remark that in section 5.7, in the context of scalar QED and W-S model, we discussed the behavior of certain energy momentum tensor (defined there and also in section 3.6 and 4.3.5), near the nontrivial fixed point of the theory. We found that in these theories as well, there is a possibility of the Green's functions of this energy momentum tensor diverging near the fixed point. We know that W-S model is a well tested model and it contains Higgs sector in it. Hence if the gravity couples to the Higgs scalars through such an energy momentum tensor, then in this model also, we may get the nontrivial results for the high energy behavior of the scattering crosssections for the processes, we are considering [28]. We may look upon $\lambda\phi^4$ theory as a toy model that we have considered to obtain such a behavior of the cross-sections. We shall now go to the calculations of the scattering crosssections, the preliminaries having already been discussed in chapter 5.

6.2 Scattering of scalars in presence of an external gravitational field.

In order that Eq. (5.63) can be applied to physical processes, it is necessary that $e^t p_1$ is on-shell for any t. This is, of course, possible only when $p_i^2 = 0$, i.e., when we are

dealing with massless scalars. In dealing with on-shell Green's functions of $\theta^{imp}_{\mu\nu}$ with massless scalars, the question of infrared divergences arises. A cross-section contains the Green's functions $G^{(n)t}_{OS}$ with external lines renormalized on shell. These differ from the Green's functions renormalized in the MS scheme as

$$G^{(n)t}_{OS} = G^{(n)t}_{MS} \left(\frac{Z}{Z'}\right)^{n/2}$$

where Z' is the on shell renormalization constant. The on-shell renormalization constant is not defined for a massless scalar due to infrared divergences. We shall, however, be interested primarily in <u>ratios</u> of differential cross-sections. As will be seen later, in these ratios, the factor $\left(\frac{Z}{Z'}\right)^{n/2}$ cancells out. For this purpose, the infrared divergences in $G^{(n)t}_{MS}$ become relevant. To this end, we would like to state that we have examined the two-point and the four-point functions of the dominant terms in $\theta^{imp}_{\mu\nu}$ [viz. $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\ \partial^2)\phi^2$] upto three loops and found no infrared on-shell divergence in $G^{(n)t}_{MS}$; n=2,4 (except for a set of exceptional external momenta). In the following, we shall assume that $G^{(n)t}_{MS}$ are calculated for <u>zero</u> <u>mass</u> (in which case m $=\bar{m}(t)=0$)and $\left(\frac{Z}{Z'}\right)$ is calculated by putting a small <u>infrared</u> <u>cutoff</u>.

Our results can be used for massive scalars provided one is at an energy scale far greater than the mass of the scalar m_p. To see this, we let $e^t p_i$, for a particular t, on-shell. Then $p_i^2 = e^{-2t}\ m_p^2$ and the right hand side of Eq. (5.63) contains a Green's function $G^{(n)t}_{\mu\nu}$ $(p_i,\ \lambda^*,\ \bar{m}(t)e^{-t},\ \mu)$ evaluated at an off-shell momentum $p_i^2 = e^{-2t}\ m_p^2$ and with the mass parameter replaced by

$\bar{m}(t)e^{-t} \simeq e^{[\gamma_m(\lambda^*)-1]t}m$ [20]. We, now, wish to discuss the extrapolation of $G^{(n)t}_{\mu\nu MS}$ to on-shell momenta. To this end, we note that if $2\gamma_m(\lambda^*) > 0$, as pointed out, in the last chapter the operator $(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\phi^2$ dominates and

$$G^{(n)t}_{\mu\nu}(p_i, \lambda^*, \bar{m}(t)e^{-t}, \mu) \propto (q_\mu q_\nu - \eta_{\mu\nu}q^2)\, G^{(n)t}_7(p_i, \lambda^*, \bar{m}(t)e^{-t}, \mu) \tag{6.2}$$

Now, we consider the cases n=2 and n=4. $G^{(2)t}_7(p_i, \lambda^*, \bar{m}(t)\, e^{-t}, \mu)$ is a Lorentz scalar and hence a function of $q^2, p_1{}^2$ and $p_2{}^2$ and is being evaluated at $p_1{}^2 = p_2{}^2 = e^{-2t}m_p{}^2$. Now, if $q^2 \gg m^2$, the extrapolation in $p_i{}^2$ from $e^{-2t}\, m^2$ to m^2 is likely to be smooth provided $G^{(2)}_7(q^2, p_1{}^2, p_2{}^2)$ has no infrared divergences as $p_1{}^2 \to 0$ and/or $p_2{}^2 \to 0$; i.e. no infrared divergences are present in the massless case. By the same argument, one could extrapolate smoothly from the mass argument $\bar{m}(t)e^{-t}$ to m provided

$$\frac{q^2}{m^2} \gg \max\left\{\left[\frac{\bar{m}(t)e^{-t}}{m}\right]^2, \left[\frac{m}{\bar{m}(t)e^{-t}}\right]^2\right\}.$$

Hence, for $q^2 \gg m^2$, one could replace

$$G^{(2)t}_7(q^2, p_1{}^2, p_2{}^2, \lambda^*, \bar{m}(t)e^{-t}, \mu) = G^{(2)t}_7(q^2, e^{-2t}m_p{}^2, e^{-2t}m_p{}^2, \bar{m}(t)e^{-t}, \lambda^*, \mu)$$

$$\simeq G^{(2)t}_7(q^2, m_p{}^2, m_p{}^2, m, \lambda^*, \mu) \tag{6.3}$$

Thus, from Eq. (6.3)

$$G^{(2)t}_{\mu\nu}(p_i, \lambda^*, \bar{m}(t)e^{-t}, \mu) \simeq (\mathrm{const})(q_\mu q_\nu - \eta_{\mu\nu}q^2)G^{(n)t}_7(p_i, \lambda^*, \bar{m}(t)e^{-t}, \mu)$$

$$\simeq (\text{Const})(q_\mu q_\nu - \eta_{\mu\nu} q^2)\ G_7^{(n)t}(q^2, \lambda^*, m, \mu)\Big|_{\text{on-shell}}$$

$$\simeq G_{\mu\nu}^{(2)t}\ (q^2,\ \lambda^*,\ m,\ \mu)\Big|_{\text{on-shell}} \qquad (6.4)$$

One thus has, from Eq. (5.63)

$$G_{\mu\nu}^{(2)t}\ (e^t q,\ \lambda,\ m,\ \mu)\Big|_{\text{on-shell}} = e^{[4-2+2\gamma_m(\lambda^*)]t} \times G_{\mu\nu}^{(2)t}\ (q,\ \lambda^*, m, \mu)\Big|_{\text{on-shell}} \qquad (6.5)$$

A similar argument can be given for the four-point function. In Eq. (6.2),

$$G_7^{(4)t}(p_i, \lambda^*, \bar{m}\ e^{-t}, \mu) \simeq G_7^{(4)t}\left[\left\{p_i \cdot p_j (i \neq j), p_i^{\ 2} (i=1,2,3,4), \lambda^*, \bar{m}\ e^{-t}, \mu\right]\right.$$

$$= G_7^{(4)t}\left[\left\{p_i \cdot p_j (i \neq j)\right\},\ p_i^{\ 2} = e^{-2t}\ m_p^{\ 2},\ \lambda^*,\ \bar{m}\ e^{-t},\ \mu\right] \qquad (6.6)$$

Now for $p_i \cdot p_j\ (i \neq j) \gg m^2$ the extrapolation in $p_i^{\ 2}$ from $e^{-2t}\ m^2$ to m^2 and in $\bar{m}(t)e^{-t}$ to m will be smooth provided there is no infrared singularity in $G_7^{(4)t}$ in the massless case. Hence, as in the case of n=2,

$$G_{\mu\nu}^{(4)t}\ \left[e^t q,\ e^{2t}\ p_i \cdot p_j (i \neq j),\ \lambda\ ,\ m,\ \mu\right]\Big|_{\text{on-shell}}$$

$$\simeq e^{2\gamma_m(\lambda^*)t}\, G^{(4)t}_{\mu\nu}\left[q,\ p_i . p_j (i\neq j),\ \lambda^*, m, \mu\right]\Bigg|_{\text{on-shell}}$$

(6.7)

(A) The scattering of a scalar by a time-independent gravitational field:

The scattering of a single massless scalar by a weak external time-independent gravitational field can be treated along the lines of same for Coulomb scattering [See e.g. Ref. 24]. We assume that the system is enclosed in a volume V in which a weak gravitational field is present. We define

$$\tilde{h}_{\mu\nu}(\vec{q}) \equiv \int_V d^3x\ e^{-i\vec{q}.\vec{x}}\ h_{\mu\nu}(\vec{x}) \tag{6.8}$$

One then has [24]

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}\left|\tilde{h}^{\mu\nu}(\vec{q})\ G^{(2)t}_{\mu\nu}\ (q, p_1^2 = o,\ \lambda,\ \mu)\right|^2 \tag{6.9}$$

where p_1 = four-momentum of the incoming scalar

where p = four-momentum of the outgoing scalar

$$q = p_1 - p_2$$

$$p_{20} - p_{10} = q_o = 0 \quad , \quad \vec{q} = \vec{p}_2 - \vec{p}_1$$

Now, we consider $\frac{d\sigma}{d\Omega}$ as a function of q^2 :

$$\frac{d\sigma}{d\Omega}(e^{2t}q^2) = \frac{1}{4\pi^2 \times 16}\left|\ \tilde{h}^{\mu\nu}(e^t\vec{q})\ G^{(2)t}_{\mu\nu}\ (e^t\vec{q}, p_i^{\ 2} = 0,\ \lambda, \mu)\right|^2 \tag{6.10}$$

Using the result of the Eq. (5.63) one obtains

$$\frac{d\sigma}{d\Omega}\,(e^{2t}q^2) = \frac{1}{4\pi^2\times 16}\, e^{4\,[1+\gamma_m\,(\lambda^*)]t}$$

$$\times \left|\tilde{h}^{\mu\nu}\,(e^t\vec{q})\; G^{(2)t}_{\mu\nu}\,(\vec{q},\, p_1{}^2 = 0,\, \lambda^*,\mu)\right|^2 \qquad (6.11)$$

The exact behavior of $\frac{d\sigma}{d\Omega}$ as a function of t depends on the behavior of $\tilde{h}^{\mu\nu}\,(e^t\vec{q})$ as a function of t, i.e. on the details of the gravitational potential. We shall study this in the concrete examples of the gravitational field due to a black hole and a neutron star in the next section. Here, we only wish to bring out the presence of a factor $e^{4\gamma_m(\lambda^*)t}$ arising from our energy-momentum tensor irrespective of a specific $h_{\mu\nu}(\vec{x})$. This factor could give rise to a large significant deviation from the otherwise expected high energy behavior if 4 $\gamma_m(\lambda^*)$ is significant. In particular, if such a deviation is detected and measured, it could actually provide the value of $[\gamma_m(\lambda^*]$ without knowing the fixed point.

An explicit application of Eq. (6.11) to black hole and neutron stars is worked out in the next section, where the complete t-dependence is obtained in that context.

(B) Scattering of two scalars in presence of a time-independent gravitational field:

The scattering process in presence of a gravitational field we are dicussing is

$$\phi(p_1) + \phi(p_2) \xrightarrow{h_{\mu\nu}(q)} \phi(q_1) + \phi(q_2)$$

with

$$q_1 + q_2 = p_1 + p_2 + q \tag{6.12}$$

where ϕ denotes a scalar particle. This scattering process is to be compared with the process in absence of gravity, i.e. solely due to $\lambda\phi^4$ interaction.

$$\phi(p_1) + \phi(p_2) \longrightarrow \phi(q_1) + \phi(q_2')$$

with

$$p_1 + p_2 = q_1 + q_2' \tag{6.13}$$

In presence of a given external field $h_{\mu\nu}(\vec{x})$, the momentum $\vec{q}$ injected is a variable and hence for a fixed p_1, p_2 and q_1, q_2 of Eq.(6.12) is a variable. In Eq. (6.13), q_2' is fixed. Hence to compare such processes we consider

$\frac{d\sigma}{d\Omega_1}$ = the differential scattering cross-section per unit solid angle that a final state scalar is detected in a solid angle $d\Omega_1$ with the second scalar not detected.

(6.14)

This quantity is defined for both the processes and hence it can be compared for both the processes.

The result for the differential scattering cross-section in the case of scattering in absence of an external gravitational field is

$$\frac{d\sigma}{d\Omega_1} = \frac{q_{10} \left| G^{(4)t} (p_1, p_2, q_1, q_2 = p_1 + p_2 - q_1) \right|^2}{128\, \pi^2 (p_{10} + p_{20}) p_{10}\, p_{20}} \tag{6.15}$$

where $G^{(4)t}$ is the four-point truncated connected Green's function of the $\lambda\phi^4$ theory. The scaling law for $G^{(4)}$ is known :

$$G^{(4)t}\ (e^t\ p_i, \lambda, \mu) \simeq G^{(4)t}\ (p_i, \lambda^*, \mu) \tag{6.16}$$

We compare $\frac{d\sigma}{d\Omega}$ for momenta p_i and $e^t p_i$ (massless case) and obtain

$$\frac{d\sigma}{d\Omega_1}(t) = e^{-2t}\ \frac{d\sigma}{d\Omega_1}(o) \tag{6.17}$$

The e^{-2t} factor is dictated by the dimensions of $\frac{d\sigma}{d\Omega_1}$.

Next we consider the process in presence of gravity. We obtain (see Appendix G),

$$S_{fi} = \frac{(2\pi)^4\ \delta^4(q_1+q_2-p_1-p_2)\ G^{(4)t}(p_1,p_2,q_1,q_2)}{4\ V^2 \sqrt{q_{10}\ q_{20}\ p_{10}\ p_{20}}}$$

$$+ \frac{\pi i}{V^2}\ \frac{\tilde{h}^{\mu\nu}(\vec{q}_1+\vec{q}_2-\vec{p}_1-\vec{p}_2)\ G^{(4)t}_{\mu\nu}(p_1,p_2,q_1,q_2)\ \delta(p_{10}+p_{20}-q_{10}-q_{20})}{4\sqrt{q_{10}q_{20}p_{10}p_{20}}}$$

$$+ \ O(h^2) \tag{6.18}$$

For the considerations to which we restrict ourselves in this work, the $O(h^2)$ term in Eq. (6.18) will not be needed. This point shall be clarified shortly. We obtain,

$$\frac{d\sigma^G}{d\Omega_1} = \frac{d\sigma}{d\Omega_1} + \frac{d\sigma^{(1)}}{d\Omega_1} + \frac{d\sigma^{(2)}}{d\Omega_1} + \frac{d\sigma^{(3)}}{d\Omega_1} \tag{6.19}$$

Here $\frac{d\sigma}{d\Omega_1}$ is the differential scattering cross-section in absence of gravity [See Eq. (6.15)] $\frac{d\sigma^{(1)}}{d\Omega_1}$ is the O(h) contribution to the differential scattering cross-section and is given by

$$\frac{d\sigma^{(1)}}{d\Omega_1} = -\frac{1}{4\times 32\pi^2 V} \frac{q_{10}\ \tilde{h}^{\mu\nu}\ (o)}{p_{10}p_{20}(p_{10}+p_{20})} \text{Im}\left[G^{(4)t}_{\mu\nu}\ (p_1,p_2,q_1,q_2)\right.$$

$$\left. \times\ G^{(4)t*}\ (p_1,p_2,q_1,q_2)\right] \qquad (6.20)$$

where it is understood that $\vec{q}_2 = \vec{p}_1 + \vec{p}_2 - \vec{q}_1$, $q_{20} = |\vec{q}_2|$ and $q_{10} = |\vec{q}_1| = p_{10} + p_{20} - |\vec{p}_1 + \vec{p}_2 - \vec{q}_1|$ (i.e. in short the four momentum conservation constraints) in evaluating $G^{(4)t}$ and $G^{(4)t}_{\mu\nu}$. $\frac{d\sigma^{(2)}}{d\Omega_1}$ is the $O(h^2)$ contribution coming entirely from the O(h) term in S_{fi}. It is given by,

$$\frac{d\sigma^{(2)}}{d\Omega_1} = \frac{1}{16\times 4\times 128\ \pi^5\ V\ p_{10}p_{20}} \int dq_{10}\ \ q_{10}\ (p_{10}+p_{20}-q_{10})d\Omega_2$$

$$\times\ |\tilde{h}^{\mu\nu}\ (\vec{q})\ G^{(4)t}_{\mu\nu}\ (p_1,p_2,q_1,q_2)|^2 \qquad (6.21)$$

with $\vec{q} \equiv \vec{q}_1 + \vec{q}_2 - \vec{p}_1 - \vec{p}_2$ and $q_{20} = (p_{10} + p_{20} - q_{10})$.

Looking at equation (6.21), we may tend to conclude that this result would be zero for the infinite volume case. It may not be so, as we have discussed in appendix G.

In addition to $\frac{d\sigma^{(2)}}{d\Omega_1}$, there is another $O(h^2)$ contribution to $\frac{d\sigma^G}{d\Omega_1}$ and this comes from the interference of the $O(h^o)$ and $O(h^2)$ terms in S_{fi} of Eq. (6.18). These two $O(h^2)$ contributions to

$\frac{d\sigma^{G}}{d\Omega_1}$ are distinguishable from each other due to the fact that in $\frac{d\sigma^{(2)}}{d\Omega_1}$ contribution comes mainly from processes in which gravity injects a nonzero momentum $\vec{q} = \vec{q}_1 + \vec{q}_2 - \vec{p}_1 - \vec{p}_2$ into the system whereas for $\frac{d\sigma^{(3)}}{d\Omega_1}$ the contribution comes strictly from momentum conserving processes i.e. $q_1 + q_2 - p_1 - p_2 = 0$. These, therefore can be distinguished from each other by applying an experimental cut. The cut to be applied can be spelt out most easily if the frame we are working with, also happens to be the C.M. frame ($\vec{p}_1+\vec{p}_2 = 0$), but can also be given in any frame. In the C.M. frame, in the momentum conserving case

$$|\vec{p}_1| = |\vec{p}_2| = |\vec{q}_1| = |\vec{q}_2|$$

Hence if one obtains $\left.\frac{d\sigma^{(G)}}{d\Omega_1}\right|_{cut}$ in which $\left|\,|\vec{q}_1|-|\vec{p}_1|\,\right| > \epsilon$ where ϵ is is small range consistent with experimental limitations, all the momentum conserving contributions $\frac{d\sigma}{d\Omega_1}$, $\frac{d\sigma^{(1)}}{d\Omega_1}$, $\frac{d\sigma^{(3)}}{d\Omega_1}$ are eliminated and one finds

$$\left.\frac{d\sigma(G)}{d\Omega_1}\right|_{cut} = \left.\frac{d\sigma^{(2)}}{d\Omega_1}\right|_{cut} \simeq \frac{d\sigma^{(2)}}{d\Omega_1} . \tag{6.22}$$

Hence, it is meaningful to study the t - dependence of $\frac{d\sigma^{(2)}}{d\Omega_1}$ alone.

Next, we shall consider the scaling properties of $\frac{d\sigma^{(1)}}{d\Omega_1}$ and $\frac{d\sigma^{(2)}}{d\Omega_1}$, that of $\frac{d\sigma}{d\Omega_1}$ being already discussed in Eq. (6.17). $\frac{d\sigma^{(1)}}{d\Omega_1}$ involves $G^{(4)t}_{\mu\nu}$ at <u>zero</u> momentum q. At $q = 0$, $G^{(4)t}_{\mu\nu}$ does not receive contribution from the last term in Eq. (5.59) which would

have lead to an effective nonzero anomalous dimension. Hence at q = 0, $G^{(4)t}_{\mu\nu}$ scales as $G^{(4)t}$ does and hence $\frac{d\sigma^{(1)}}{d\Omega_1}$ scales as $\frac{d\sigma}{d\Omega_1}$, i.e.

$$\frac{d\sigma^{(1)}}{d\Omega_1}(t) = e^{-2t} \frac{d\sigma}{d\Omega_1}(o) \tag{6.23}$$

The scaling property of $\frac{d\sigma^{(2)}}{d\Omega_1}$ is more involved because of its dependence on $\tilde{h}^{\mu\nu}(\vec{q})$ and we cannot obtain an explicit scaling law except by considering special cases as done in the next section. To this end, we make a change of integration variable $q_1 = e^t \bar{q}_1$, (which includes $q_{10} = e^t \bar{q}_{10}$) in Eq. (6.21). One then has ($q = \bar{q}_1 + q_2 - p_1 - p_2$),

$$\frac{d\sigma^{(2)}}{d\Omega 1}(t) = \frac{e^t}{16\times 4\times 128\pi^5 \, V \, p_{10}p_{20}} \int d\bar{q}_{10}\bar{q}_{10}\left[p_{10} + p_{20} - \bar{q}_{10}\right] d\Omega_2$$

$$\times \left|\tilde{h}^{\mu\nu}(e^t\vec{q})G^{(4)t}_{\mu\nu}(e^t p_1, e^t p_2, e^t \bar{q}_1, e^t q_2)\right|^2$$

$$\simeq e^{[1+4\gamma_m(\lambda^*)]t} \times \frac{1}{16\times 4\times 128\,\pi^5 \, V \, p_{10}\, p_{20}}$$

$$\times \int d\bar{q}_{10}\, \bar{q}_{10}\left[p_{10}+p_{20}-\bar{q}_{10}\right]d\Omega_2 \left|\tilde{h}^{\mu\nu}\left[e^t\vec{q}\right] G^{(4)t}_{\mu\nu}\left[p_1, p_2, \bar{q}_1, q_2, \lambda^*, \mu\right]\right|^2$$

$$\tag{6.24}$$

The t-dependence now appears, firstly in the form of an overall factor, which in particular contains a factor of $e^{4\gamma_m(\lambda^*)t}$ and secondly through $\tilde{h}^{\mu\nu}(e^t\vec{q})$. As noted in the case of the two point

function, the factor of $e^{4\gamma_m(\lambda^*)t}$ is over and above the expected behavior for an operator $\theta_{\mu\nu}^{imp}$ of zero anomalous dimension and can, in fact, give rise to a large multiplicative contribution. To extract the exact t-dependence, the form of $\tilde{h}^{\mu\nu}(e^t\vec{q})$ has to be known and this depends on the details of the gravitational field. In the next section, we shall consider a special case of the gravitational field produced by a black hole or a neutron star where a further simplification of Eq. (6.24) can be made.

6.3 Applications to concrete examples

In this section, we shall apply the results developed in the previous section to the concrete examples of gravitational fields such as those produced by a black hole or a neutron star.

We assume that the metric is static isotropic and use the Schwarzschild form of the metric [25].

$$\begin{aligned} d\tau^2 &= \left[1 - \frac{r_o}{r}\right]dt^2 - \left[1 - \frac{r_o}{r}\right]^{-1}dr^2 - \left(r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2\right) \\ &= \left[1 - \frac{r_o}{r}\right]dt^2 - \left\{\left[1-\frac{r_o}{r}\right]^{-1} - 1\right\}\frac{x_i x_j}{r^2}\, dx^i dx^j - dx_i dx^i \end{aligned} \tag{6.25}$$

with $r_o = 2GM$

Hence

$$g_{oo} = 1 - \frac{r_o}{r} \quad ; \; h_{oo} = -\frac{r_o}{r}$$

$$g_{ij} = \eta_{ij} - \left\{\left[1 - \frac{r_o}{r}\right]^{-1} - 1\right\} \frac{x_i x_j}{r^2}$$

$$h_{ij} = - \left\{[1 - \frac{r_o}{r}]^{-1} - 1\right\} \frac{x_i x_j}{r^2}$$

$$h_\mu{}^\mu = h_o{}^o + h_i{}^i = h_{oo} - h_{ii}$$

$$= - \frac{r_o}{r} + \left\{(1 - \frac{r_o}{r})^{-1} - 1\right\} \qquad (6.26)$$

Now it is possible to show that for $qr_o >> 1$, $h^{\mu\nu}(\vec{q})$ (when nonzero) is of $O(\frac{r_o^2}{q})$ for all μ and ν. (Special cases will be dealt with below). Hence in $\tilde{h}^{\mu\nu}(\vec{q})\ G_{\mu\nu}^{(n)t}$, the dominant contribution comes from that term in $G_{\mu\nu}^{(n)t}$ which is large viz the last term in Eq. (5.59) provided $q \neq 0$ and λ is sufficiently close to λ^*. Hence we shall evaluate only this contribution to $\tilde{h}^{\mu\nu}(\vec{q})G_{\mu\nu}^{(n)t}$. In other words,

$$\tilde{h}^{\mu\nu}(\vec{q})\ G_{\mu\nu}^{(n)t}(p_i) \simeq \frac{\beta(\lambda)}{24\gamma_m(\lambda)} \frac{dK(\lambda)}{d\lambda} \tilde{h}^{\mu\nu}(\vec{q})(q_\mu q_\nu - q^2\eta_{\mu\nu})G_7^{(n)t}(p_i) \qquad (6.27)$$

This involves the contribution $(q_\mu q_\nu - \eta_{\mu\nu}q^2)\ \tilde{h}^{\mu\nu}(\vec{q})$ which we shall evaluate below:

$$(q^\mu q^\nu - \eta^{\mu\nu}\ q^2)\ \tilde{h}_{\mu\nu}\ (\vec{q})$$

$$= (q^\mu q^\nu - \eta^{\mu\nu}\ q^2) \int h_{\mu\nu}\ (\vec{x})\ \ e^{-i\ \vec{q}.\vec{x}}\ d^3x \qquad (6.28)$$

Noting that $q^o = 0$, and using the expressions for $h_{\mu\nu}(\vec{x})$ of Eq. (6.26), this simplifies to

$$q_i q_j \frac{\partial}{\partial q_i}\frac{\partial}{\partial q_j} f(q^2) + q^2 \left[c(q^2) - d(q^2)\right] \tag{6.29}$$

where

$$f(q^2) \equiv \int \frac{1}{r^2}\left\{\left[1 - \frac{r_o}{r}\right]^{-1} - 1\right\} e^{-i\vec{q}.\vec{x}}\, d^3x$$

$$c(q^2) \equiv \int \left\{\left[1 - \frac{r_o}{r}\right]^{-1} - 1\right\} e^{-i\vec{q}.\vec{x}}\, d^3x$$

$$d(q^2) \equiv \int \frac{r_o}{r}\, e^{-i\vec{q}.\vec{x}}\, d^3x \tag{6.30}$$

Thus one has,

$$(q^\mu q^\nu - \eta^{\mu\nu} q^2)\, \tilde{h}_{\mu\nu}(\vec{q}) = 2\vec{q}^2 f'(\vec{q}^2) + 4(\vec{q}^2)^2 f''(\vec{q}^2) - \vec{q}^2 d(q^2) + \vec{q}^2 c(\vec{q}^2) \tag{6.31}$$

Now using

$$\int F(r)\, e^{-i\vec{q}.\vec{x}}\, d^3x = \frac{4\pi}{|\vec{q}|} \int F(r)\, r \sin qr\, dr \tag{6.32}$$

one obtains

$$f(q^2) = \frac{4\pi}{|\vec{q}|} \int_{x_1}^{x_2} \frac{\sin qr_o x}{x\,(x-1)}\, dx$$

$$c(q^2) = \frac{4\pi\, r_o^2}{|\vec{q}|} \int_{x_1}^{x_2} \frac{x \sin qr_o x}{(x-1)}\, dx$$

$$d(q^2) = \frac{4\pi\, r_o^2}{|\vec{q}|} \int_{x_1}^{x_2} \sin q\, r_o x\, dx \tag{6.33}$$

where we have enclosed the system in a spherical shell of radii r_1

and r_2 ($r_1 > r_o$, $r_2 > r_1 > r_o$) and have set $r = r_o x$.

Using these expressions in Eq. (6.31) we obtain, after some simplifications,

$$\left[q^\mu q^\nu - \eta^{\mu\nu} q^2\right] \tilde{h}_{\mu\nu}(\vec{q}) = \vec{q}^{\,2} \left\{ - \frac{8\pi r_o}{|\vec{q}|^2} \int_{x_1}^{x_2} \frac{\cos q r_o x}{x-1}\, dx \right.$$

$$+ \frac{8\pi}{|\vec{q}|^3} \int_{x_1}^{x_2} \frac{\sin q r_o x}{x\,(x-1)}\, dx$$

$$\left. - \frac{4\pi r_o^2}{|\vec{q}|} \int_{x_1}^{x_2} \sin(q r_o x) dx \right\} \qquad (6.34)$$

The first term in the curly bracket will be shown to be of $O\left(\frac{1}{|\vec{q}|^3}\right)$ in the Appendix H. By a similar technique the second term is seen to be of $O\left(\frac{1}{r_o|\vec{q}|^4}\right)$. Hence for $q r_o \gg 1$, the last term in the curly bracket dominates. It is

$$- \frac{4\pi r_o^2}{|\vec{q}|} \int_{x_1}^{x_2} \sin(q r_o\, x) dx = \frac{4\pi r_o}{q^2} \left[\cos(q r_2) - \cos(q r_1)\right] \qquad (6.35)$$

Now, we consider the applications of this result to the scattering of a scalar by an external gravitational field and scattering of two scalars in presence of an external gravitational field considered above. In either case, we shall assume that the system is confined within a spherical shell of radii r_1 and r_2 ($> r_o$).

(A) Scattering of a scalar by a gravitational field:

First, consider the scattering of a scalar by the gravitational field described by $h_{\mu\nu}(\vec{x})$ of Eq. (6.26). Eq.(6.9) involves the quantity $\tilde{h}^{\mu\nu}(\vec{q})\, G^{(2)t}_{\mu\nu}(q^2, p_i^{\,2} = 0, \lambda, \mu)$.
From Eq. (5.59)

$$G^{(2)t}_{\mu\nu} \simeq \frac{\beta(\lambda)}{24\gamma_m(\lambda)} \frac{dK(\lambda)}{d\lambda} (q_\mu q_\nu - q^2\eta_{\mu\nu})\, G^{(2)t}_7 (q^2, p_i^{\,2}=0, \lambda, \mu)$$

Using Eq. (6.34) and (6.35) one obtains,

$$\tilde{h}^{\mu\nu}(\vec{q})\, G^{(2)t}_{\mu\nu} \simeq \frac{\beta(\lambda)}{24\gamma_m(\lambda)} \frac{dK(\lambda)}{d\lambda} (4\pi r_o)[\cos(qr_2) - \cos(qr_1)]G^{(2)t}_7 \tag{6.36}$$

Hence,

$$\frac{d\sigma}{d\Omega} = \frac{r_o^2}{144\times16}\left[\frac{\beta(\lambda)}{\gamma_m(\lambda)}\frac{dK(\lambda)}{d\lambda}\right]^2\left[\cos(qr_2)-\cos(qr_1)\right]^2\left|G^{(2)t}_7(q^2, p_i^{\,2}=0, \lambda, \mu)\right|^2 \tag{6.37}$$

Now

$$G^{(2)t}_7(e^{2t}q^2, e^{2t}p_i^{\,2} = 0, \lambda, \mu) \simeq e^{2[\gamma_m(\lambda^*)]t}\, G^{(2)t}_7(q^2, p_i^{\,2} = 0, \lambda^*, \mu) \tag{6.38}$$

Thus,

$$\frac{d\sigma}{d\Omega}(t) = e^{4[\gamma_m(\lambda^*)]t}\left[\cos(e^t qr_2) - \cos(e^t qr_1)\right]^2$$
$$\times \frac{r_o^2}{16\times144}\left[\frac{\beta(\lambda)}{\gamma_m(\lambda)}\frac{dK(\lambda)}{d\lambda}\right]^2\left|G^{(2)t}_7(q^2, p_i^{\,2} = 0, \lambda^*, \mu)\right|^2 \tag{6.39}$$

Now as t varies, the factor $[\cos(e^t qr_2) - \cos(e^t qr_1)]$ oscillates between 0 and 4, and does not show an overall variation. The overall variation is provided by the factor $\exp\left\{4\gamma_m(\lambda^*)t\right\}$.

(B) Scattering of two scalars in presence of gravity:

Next, we turn to scattering of two scalars in presence of an external gravity. Eq. (6.24) gives,

$$\frac{d\sigma^{(2)}}{d\Omega_1} = \frac{1}{512\pi^5 \; V \; p_{10}p_{20}\times 16} \; e^{[1+4\gamma_m(\lambda^*)]t}$$

$$\times \int d\bar{q}_{10}\bar{q}_{10}(p_{10}+p_{20} - \bar{q}_{10})d\Omega_2 \left|\tilde{h}^{\mu\nu}(e^t\vec{q})G^{4)t}_{\mu\nu}(p_1, p_2, \bar{q}_1, q_2, \lambda^*, \mu)\right|^2 \tag{6.40}$$

This involves

$$\tilde{h}^{\mu\nu} \; (e^t\vec{q}) \; G^{4)t}_{\mu\nu} \; (p_1, p_2, \bar{q}_1, q_2, \lambda^*, \; \mu) \tag{6.41}$$

Unlike the case of the two-point function, in Eq. (6.41) there is an integration over $\bar{q}_{10}$ which involves the variable $|\vec{q}|$. Now, from Eq. (6.34), the term of $O(\frac{r_o}{|\vec{q}|^2})$ dominates the term of $O(\frac{1}{|\vec{q}|^3})$ even for $q = |\vec{q}| >> \frac{1}{r_o}$. If r_o is of the order of 1 km, this implies that one may use the large $|\vec{q}|$ approximation for $\tilde{h}^{\mu\nu}$ $(\vec{q})$ even from very small $|\vec{q}|$. Hence as in the case of the two point function,

$$\tilde{h}^{\mu\nu} \; (e^t \; \vec{q}) \; G^{(4)t}_{\mu\nu} \; (p_1, p_2, \bar{q}_1, q_2, \lambda^*, \mu)$$

$$\simeq e^{-2t} 4\pi r_o [\cos(e^t qr_2) - \cos(e^t qr_1)]$$

$$\times \frac{\beta(\lambda)}{24\,\gamma_m(\lambda)} \frac{dk(\lambda)}{d\lambda} \quad G_7^{(4)t}(p_1, p_2, \bar{q}_1, q_2, \lambda^* \mu) \tag{6.42}$$

Thus,

$$\frac{d\sigma^{(2)}}{d\Omega_1}(t) = \frac{r_o^2}{32\pi^3 \times 16} e^{[4\gamma_m(\lambda^*)-3]t} \left[\frac{\beta(\lambda)}{24\gamma_m(\lambda)}\frac{dK(\lambda)}{d\lambda}\right]^2$$

$$\times \int \frac{\bar{q}_{10} d\bar{q}_{10}\ (p_{10}+p_{20}-\bar{q}_{10})}{p_{10}p_{20}\ V}\left[\cos(e^t qr_2) - \cos(e^t qr_1)\right]^2$$

$$\times \left| G_7^{(4)t}\ (p_1, p_2, \bar{q}_1, q_2, \lambda^*, \mu)\right|^2 d\Omega_2 \tag{6.43}$$

$$= e^{[4\gamma_m((\lambda^*)-3]t} \frac{d\sigma^{(2)}}{d\Omega_1}(t = 0) \tag{6.44}$$

where we have replaced $[\cos(e^t qr_2) - \cos(e^t qr_1)]^2$ by its average value both for $t \neq 0$ and $t = 0$ [in view of the fact that qr_i are large and $\cos(qr_i e^t)$ are rapidly oscillating functions both for $t = 0$ and $t \neq 0$]. As mentioned in the previous section, $\frac{d\sigma^{(2)}}{d\Omega_1}$ can be obtained by applying a suitable cut and will show the t-dependence given by Eq. (6.44), which involves $4\gamma_m(\lambda^*)$.

(C) Numerical estimates:

We now make a numerical estimate for $\frac{d\sigma^{(2)}}{d\Omega_1}$ as it has an interesting high energy behavior. At ordinary energies it is expected to be small but as it has a different dependence on t as compared to that of $\frac{d\sigma}{d\Omega_1}$, it can lead to a significant corrections at sufficiently high energies as seen from the estimates given

below:

In Eq. (5.59), the last term dominates provided μ (and also the energy scale of momenta) is high enough so that $\lambda \simeq \lambda^*$ and $\alpha = 2\gamma_m(\lambda^*) > 0$. Then the coefficient $\frac{\beta(\lambda)}{24\gamma_m(\lambda)} \frac{dK(\lambda)}{d\lambda}$ must (and can) become large as compared to that of other operators e.g. $G_5^{(n)}$. Thus let $\frac{\beta(\lambda)}{24\gamma_m(\lambda)\times 4} \frac{dK(\lambda)}{d\lambda} \simeq 100$. Let, for concreteness, the mass scale $\mu \simeq 100$ GeV, and let $p_{10} = q_{10} = 100$ GeV defining the energy scale. On dimensional grounds, $G_7^{(4)}$ $(p_1, p_2, \bar{q}_1, q_2) \sim \frac{1}{\bar{q}_{10}^{\,2}}$ for large $\bar{q}_{10} >> \mu$. $G_7^{(4)}$ also depends on coupling constnat λ^*. Taking $G_7^{(4)} \sim \frac{1}{\bar{q}_{10}^{\,2}}$ amounts to making an implicit assumption about the magnitude of λ^*. But the following is meant to be a crude estimate. Let $\langle [\cos(e^t qr_1)]^2 \rangle \simeq 2$. We approximate $\int \frac{q_{10}\, q_{20}\, dq_{10}}{\bar{q}_{10}^{\,4}} \sim \frac{1}{\mu}$. For concreteness we let $r_2 = 4r_o$ and $r_1 = 3r_o$. We replace $\int d\Omega_2$ by 4π. We then obtain

$$\frac{d\sigma^{(2)}}{d\Omega_1}(t) = 3.5 \times \frac{1}{(r_o\mu)} \times \frac{1}{\mu^2} \times \exp \{[4\gamma_m(\lambda^*)-3]t\} \qquad (6.45)$$

Taking $r_o = 1$ km for the black hole or the neutron star, $r_o\mu \simeq 5\times 10^{20}$ and thus

$$\frac{d\sigma^{(2)}}{d\Omega_1}(t) \simeq \frac{7.2\times 10^{-21}}{\mu^2} \exp \{[4\gamma_m(\lambda^*)-3]t\} \qquad (6.46)$$

To make an estimate of $\frac{d\sigma}{d\Omega_1}$ of Eq. (6.15), we replace $|G^{(4)t}(p_1, p_2, q_1, q_2)|$ by its lowest order value at the fixed point i.e. $|G^{(4)t}| \sim \lambda^*$ which we take to be 1. Then Eq. (6.15) leads to

$$\frac{d\sigma}{d\Omega_1}(0) \simeq \frac{\mu \,.\, (\lambda^*)^2}{128\pi^2 (2\mu)\mu^2} \simeq \frac{1}{256\,\pi^2} \quad \frac{1}{\mu^2}$$

and Eq. (6.17) leads to,

$$\frac{d\sigma}{d\Omega_1}(t) = \frac{1}{256\,\pi^2} \quad \frac{1}{\mu^2} \quad e^{-2t} \tag{6.47}$$

Eqs. (6.46) and (6.47) lead to

$$\frac{d\sigma^{(2)}}{d\Omega_1}(t) \Big/ \frac{d\sigma}{d\Omega_1}(t) = 1.8 \times 10^{-17} \exp \{[4\gamma_m(\lambda^*)-1]t\} \tag{6.48}$$

Obviously, at ordinary energies this ratio is very small. But it is a very sensitive function of $\gamma_m(\lambda^*)$. Thus if $\gamma_m(\lambda^*)$ is large enough, say, $\gamma_m(\lambda^*) = 2$, then for energy scale E such that $\frac{E}{\mu} = 250$, the ratio of two cross-sections becomes one. This corresponds to an energy scale of 25 TeV, well below Planck scale, where one ordinarily expects gravity to begin to play a significant role. Even below this scale of 25 TeV, the ratio can be a substantial fraction indicating a need to take gravity into account.

To summarise, $\frac{d\sigma^G}{d\Omega_1}$ has, upto $O(h^2)$ three distinct contributions: the strong interaction contribution $\frac{d\sigma}{d\Omega_1}$, $O(h)$ contribution $\frac{d\sigma^{(1)}}{d\Omega_1}$; both of which have the same scaling law and the latter remains a small fraction of the former at all energies. But then there is also the contribution $\frac{d\sigma^{(2)}}{d\Omega_1}$ very small at ordinary energies, but which can become significant at high energy scales.

Eq. (6.48) provides a test for whether the gravity couples to a scalar field via this $\theta_{\mu\nu}^{imp}$ whose significance has been explained in Sec. 6.1. It can then also determine the magnitude of or a bound on $\gamma_m(\lambda^*)$.

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APPENDIX - A

We have defined [See Eq.(2.16)]

$$Z_e^{2q}\, Z_m^{-1} = 1 + \sum_{r=1}^{\infty} \sum_{s=1}^{r} \frac{Y_s^{qr}\, e^{2r}}{\epsilon^s} \qquad (A.1)$$

In this appendix, we shall derive the relations among Y_s^{qr} used in the text. β^e is defined by

$$\left[\beta^e(e) - \frac{1}{2}\, e\epsilon\right] = \mu \frac{\partial}{\partial\mu}\, e = \mu \frac{\partial}{\partial\mu} \left[e_o \mu^{-\epsilon/2}\, Z_e^{-1}\right]$$

$$= - \frac{e\epsilon}{2} - e\, \mu \frac{\partial}{\partial\mu} \operatorname{Ln} Z_e$$

This implies

$$\mu \frac{\partial}{\partial\mu} \operatorname{Ln} Z_e = - \frac{\beta^e(e)}{e} \qquad (A.2)$$

Further Z_m^{-1} satisfies

$$\mu \frac{\partial}{\partial\mu} \operatorname{Ln}\ Z_m^{-1} = 2\, \gamma_m(e) \qquad (A.3)$$

Thus, we have

$$\mu \frac{\partial}{\partial\mu} \operatorname{Ln} \left[Z_e^{2q}\, Z_m^{-1}\right] = \left[2\, \gamma_m(e) - 2q\, \frac{\beta^e(e)}{e} \right]$$

or equivalently

$$\mu \frac{\partial}{\partial\mu} \left[Z_e^{2q}\, Z_m^{-1}\right] = \left[\beta^e(e) - \frac{1}{2}\, e\epsilon\right] \frac{\partial}{\partial e} \left[Z_e^{2q}\, Z_m^{-1}\right]$$

$$= \left[2\, \gamma_m(e) - 2q\, \frac{\beta^e(e)}{e}\right] \left[Z_e^{2q}\, Z_m^{-1}\right] \qquad (A.4)$$

Now consider the coefficient of e^{2r+2}/ϵ^r on both sides of the Eq. (A.4). Defining

$$\beta^e(e) = \beta_3^e\, e^3 + O(e^5)$$

$$\gamma_m(e) = \gamma_{m2}\, e^2 + O(e^4) \qquad (A.5)$$

We obtain,

$$-\frac{1}{2}(2r+2)\, Y_{r+1}^{q,r+1} + \beta_3^e\, Y_r^{qr}\,(2r)$$

$$= [2\gamma_{m2} - 2q\,\beta_3^e\,]\, Y_r^{qr}$$

i.e.

$$Y_{r+1}^{q,r+1} = \frac{1}{r+1}\left[(2r+2q)\,\beta_3^e - 2\gamma_{m2}\right] Y_r^{qr} \qquad (A.6)$$

To obtain Y^{qr} using Eq. (A.6), one needs to only know Y_1^{q1}. Y_1^{q1} is easily related to β_3^e and γ_{m2}. The result is

$$Y_1^{q1} = [\,2q\,\beta_3^e - 2\gamma_{m2}\,] \qquad (A.7)$$

As seen from the values of β_3^e and γ_{m2} stated in Eq. (B.9) Y_r^{qr} is always nonzero. (Here, $q \geq 0$).

We now deduce a number of related results used in the text:

(i)
$$\begin{vmatrix} Y_4^{14} & 2Y_3^{23} \\ Y_3^{13} & 2Y_2^{22} \end{vmatrix} = 2\, Y_3^{13}\, Y_2^{22} \begin{vmatrix} \frac{1}{4}(8\beta_3^e - 2\gamma_{m2}) & \frac{1}{3}(8\beta_3^e - 2\gamma_{m2}) \\ 1 & 1 \end{vmatrix}$$

$$\neq 0$$

This has been used below Eq. (2.26).

(ii) $$\beta_3^e \; Y_1^{1o} - \frac{1}{2} \; Y_2^{o2} = Y_1^{o1} \left[\beta_3^e - \frac{1}{2} \cdot \frac{1}{2} \left(2\beta_3^e - 2 \, \gamma_{m2}\right)\right]$$

$$= Y_1^{o1} \left[\beta_3^e/2 + \frac{1}{2} \, \gamma_{m2}\right)\right] \neq 0$$

This has been used below Eq. (2.36).

APPENDIX - B

We have defined $Y = \frac{\partial}{\partial\lambda} Z_{17} \Big|_{\lambda=0} = \sum_{r=1}^{\infty} \sum_{p=1}^{\infty} \frac{\tilde{Y}_p^{\,r} (e^2)^r}{\epsilon^p}$

We wish to relate the quantities $\tilde{Y}_{2}^{2}$, $\tilde{Y}_{3}^{3}$, $\tilde{Y}_{4}^{4}$, $\tilde{Y}_5^5$ that appear in Eqs. (2.22), (2.26), (2.30), (2.31) which we wish to show are inconsistent. These relations are obtained via the renormalization group equation satisfied by Z_{17}, viz:

$$\left[-\lambda\epsilon + \beta^{\lambda}(\lambda,e)\right] \frac{\partial}{\partial\lambda} Z_{17} + \left[\frac{-e\epsilon}{2} + \beta^{e}(\lambda,e)\right] \frac{\partial Z_{17}}{\partial e} - 2\,\gamma_m Z_{17}$$

$$= Z_{11}\,\gamma_{17} + Z_{15}\,\gamma_{57} \tag{B.1}$$

We expand Z_{17} (showing only the necessary terms)

$$Z_{17} = a' \frac{\lambda e^2}{\epsilon} + b' \frac{\lambda e^4}{\epsilon^2} + c \frac{\lambda e^6}{\epsilon^3} + d \frac{\lambda e^8}{\epsilon^4} + f \frac{\lambda e^{10}}{\epsilon^5} + \frac{k e^4}{\epsilon}$$

$$+ h \frac{e^6}{\epsilon^2} + g \frac{e^8}{\epsilon^3} + m' \frac{e^{10}}{\epsilon^4} + d' \frac{\lambda^2 e^6}{\epsilon^4} + f' \frac{\lambda^2 e^4}{\epsilon^3}$$

$$+ c' \frac{\lambda^2 e^2}{\epsilon^2} + j \frac{\lambda^3 e^2}{\epsilon^3} + \ldots\ldots \tag{B.2}$$

We also expand $\beta^{\lambda}(\lambda,e)$, $\beta^{e}(\lambda,e)$, $\gamma_m(\lambda,e)$ as

$$\beta^{\lambda}(\lambda,e) = \lambda^2 \frac{\partial}{\partial\lambda} Z_{\lambda}^{(1)} + \frac{e\lambda}{2} \frac{\partial}{\partial e} Z_{\lambda}^{(1)} + \frac{e}{2} \frac{\partial}{\partial e} \delta\lambda^{(1)} - \delta\lambda^{(1)}$$

$$\equiv \beta_1^{\lambda}\, e^4 + \beta_2^{\lambda}\, \lambda e^2 + \tilde{\beta}_1^{\lambda}\, \lambda^2 + \ldots\ldots$$

$$\beta^{e}(\lambda,e) = \frac{e^2}{2}\frac{\partial}{\partial e} \quad Z_e^{(1)} + e\lambda \frac{\partial Z_e^{(1)}}{\partial \lambda} \equiv \beta_3^e \, e^3 + \ldots\ldots$$

$$\gamma_m(\lambda,e) = \gamma_{m1}\lambda + \gamma_{m2}\, e^2 + \ldots\ldots \tag{B.3}$$

Next, we compare the coefficients of $\frac{\lambda e^{10}}{\epsilon^4}$, $\frac{e^{10}}{\epsilon^3}$, $\frac{\lambda e^8}{e^3}$

$\frac{\lambda e^6}{\epsilon^2}$, $\frac{e^8}{\epsilon^2}$, $\frac{e^6}{\epsilon}$, $\frac{\lambda e^4}{\epsilon}$, $\frac{\lambda^2 e^6}{\epsilon^3}$, $\frac{\lambda^2 e^4}{\epsilon^2}$, $\frac{\lambda^3 e^2}{\epsilon^2}$ on both sides of Eq.(B.1)

and obtain successively

$$-6f + (8\beta_3^e + \beta_2^\lambda - 2\gamma_{m2})d - 2\gamma_{m1}\, m' + 2\beta_1^\lambda\, d' = 0 \tag{B.4}$$

$$\beta_1^\lambda\, c - 5m' + (8\beta_3{}^e - 2\,\gamma_{m2})\, g = 0 \tag{B.5}$$

$$-5d + (6\beta_3^e + \beta_2^\lambda - 2\,\gamma_{m2})c - 2\gamma_{m1}\, g + 2\beta_1^\lambda\, f' = 0 \tag{B.6}$$

$$-4c + (4\beta_3^e + \beta_2^\lambda - 2\,\gamma_{m2})b' - 2\gamma_{m1}\, h + 2\beta_1^\lambda\, c' = 0 \tag{B.7}$$

$$\beta_1^\lambda\, b' - 4g + (6\beta_3^e - 2\,\gamma_{m2})\, h = 0 \tag{B.8}$$

$$\beta_1^\lambda\, a' - 3h - 2\,\gamma_{m2}\, k = 0 \tag{B.9}$$

$$-3b' + (\beta_2^\lambda - 2\,\beta_3^e - 2\,\gamma_{m2})\, a' - 2\gamma_{m1} k = 0 \tag{B.10}$$

$$-5d' + 3\beta_1^\lambda j + (\tilde{\beta}_1{}^\lambda - 2\gamma_{m1})c + (2\beta_2^\lambda + 4\beta_3^e - 2\gamma_{m2})f' = 0 \tag{B.11}$$

$$-4j + (2\tilde{\beta}_1{}^\lambda - 2\gamma_{m1})c' = 0 \tag{B.12}$$

$$-4f' + (2\beta_2{}^\lambda + 2\beta_3^e - 2\gamma_{m2})c' + (\tilde{\beta}_1{}^\lambda - 2\gamma_{m1})\, b' = 0 \tag{B.13}$$

This, together with

$$- 3c' = (2\gamma_{m1} - \tilde{\beta}_1^{\lambda})\ a' \qquad (B.14)$$

obtained in I and the following values calculated explicitly,

$$\beta_1^{\lambda} = \frac{9}{8\pi^2} ; \quad \tilde{\beta}_1^{\lambda} = \frac{1}{8\pi^2} \qquad \beta_2^{\lambda} = - \frac{1}{\pi^2}$$
$$\gamma_{m1} = \frac{1}{48\pi^2} \qquad \gamma_{m2} = - \frac{3}{16\pi^2}$$
$$\beta_3^{e} = \frac{1}{48\pi^2} \qquad a' = \frac{1}{3(16\pi^2)^2} \qquad (B.15)$$

enables one to express $\tilde{Y}_{2,}^{\ 2}$ $\tilde{Y}_{3,}^{\ 3}$ $\tilde{Y}_{4,}^{\ 4}$ $\tilde{Y}_{5}^{\ 5}$ (denoted by b', c, d, f respectively in the above equations) in terms of "k", the simple pole term in Z_{17} of $O(e^4)$, a quantity which we have not calculated. This enables one to look upon equations (2.22), (2.26), (2.30) and (2.31) as four equations in three unknowns $(g_o^1\ Y_3^{13})$, $(g_2^{-1}\ Y_2^{22})$ and k and verify their inconsistency.

APPENDIX - C

In this appendix, we shall show that the physical matrix elements of $\theta^{imp}_{\mu\nu}$ are gauge-independent iff $\bar{k}$ is independent of ξ. To show this we note that

$$\theta^{imp}_{\mu\nu} = \theta^{(1)}_{\mu\nu} + \theta^{(2)}_{\mu\nu} + \theta^{(3)}_{\mu\nu} + \theta^{(4)}_{\mu\nu} \tag{C.1}$$

where

$$\theta^{(1)}_{\mu\nu} = - \eta_{\mu\nu} (\mathcal{L} + \frac{1}{2} \xi_0 (\partial.A)^2) - F_{\nu\alpha} F^{\alpha}_{\mu}$$
$$+ \frac{1}{2} \left[(D_\mu\phi)^* (D_\nu\phi) + (D_\nu\phi)^* (D_\mu\phi)\right]$$

$$\theta^{(2)}_{\mu\nu} = \xi_0 \left[\partial_\mu(\partial.A) A_\nu + \partial_\nu(\partial.A) A_\mu\right] - \eta_{\mu\nu}\xi_0 \partial^\rho(\partial.A)A_\rho$$
$$- \frac{1}{2} \eta_{\mu\nu} \xi_0 (\partial.A)^2$$

$$\theta^{(3)}_{\mu\nu} = -G (\lambda, e^2, \xi, \epsilon) (\partial_\mu\partial_\nu - \partial^2 \eta_{\mu\nu}) \phi^*\phi$$

$$\theta^{(4)}_{\mu\nu} = - \bar{k} (\lambda, e^2, \frac{m^2}{\mu^2}, \xi) (\partial_\mu\partial_\nu - \partial^2 \eta_{\mu\nu}) (\phi^*\phi)^R \tag{C.2}$$

Now $\theta^{(1)}_{\mu\nu}$ is a gauge-invariant operator. Hence its physical matrix elements (barring certain exceptional external momenta) are ξ-independent. We shall show that WT indentities imply that the physical matrix elements of $\theta^{(2)}_{\mu\nu}$ vanish ($q \neq 0$). We shall also show that $\theta^{(2)}_{\mu\nu}$ does not mix with $(\partial_\mu\partial_\nu - \partial^2 \eta_{\mu\nu}) (\phi^*\phi)$. Hence from Eqs. (3.3) and (3.5),

$$\left\{\langle\theta_{\mu\nu}\rangle\right\}^{div} = G'(\lambda, e^2, \xi, \epsilon) (\partial_\mu\partial_\nu - \partial^2 \eta_{\mu\nu}) \langle\phi^*\phi\rangle^R$$
$$= \left\{\langle \theta^{(1)}_{\mu\nu}\rangle\right\}^{div} \tag{C.3}$$

Thus $G'(\lambda, e^2, \xi, \epsilon)$ is the renormalization constant of a gauge-invariant operator $\theta^{(1)}_{\mu\nu}$ with another gauge-invariant operator $(\partial_\mu\partial_\nu - \partial^2\eta_{\mu\nu})(\phi^*\phi)$ and hence is ξ independent. Also Z_m^{-1}, the multiplicative renormalization constant of a gauge invariant operator $(\phi^*\phi)$, is also ξ-independent. Hence $G = G' Z_m$ is also ξ independent. Thus $\theta^{(3)}_{\mu\nu}$ is a gauge-invariant operator with a ξ-independent coefficient and hence has ξ-independent physical matrix elements. Thus

$$\frac{\partial}{\partial\xi} \langle\theta^{imp}_{\mu\nu}\rangle_{phy} = \frac{\partial}{\partial\xi} \langle\theta^{(4)}_{\mu\nu}\rangle_{phy}$$

$$= -\frac{\partial}{\partial\xi}\bar{k}(\lambda, e^2, \frac{m^2}{\mu^2}, \xi)(\partial_\mu\partial_\nu - \partial^2\eta_{\mu\nu})\langle\phi^*\phi\rangle^R_{phy}$$

$$= 0 \tag{C.4}$$

iff $\frac{\partial}{\partial\xi}\bar{k}(\lambda, e^2, \frac{m^2}{\mu^2}, \xi) = 0$ proving the result that $\bar{k}$ and hence k is ξ-independent.

Finally, we prove the two points regarding $\theta^{(2)}_{\mu\nu}$ mentioned below Eq. (C.2). $\theta^{(4)}_{\mu\nu}$ contributes only when $q \neq 0$. Hence, to determine the gauge independence of $\bar{k}$, it is sufficient to consider $\theta^{(2)}_{\mu\nu}$ at $q \neq 0$. Matrix elements of $\theta^{(2)}_{\mu\nu}$ can be determined from the WT identity derived in Eq. (B.4) of Ref. 14. It reads

$$\langle -\xi_0 \int d^n y\, [\partial.A.(y)]\, \theta[A(y)] + \int J_\mu(x)\, \partial^\mu_x G(x,y)\, \theta[A(y)] d^n x\, d^n y$$

$$+ \int J^*(x)\, ie_0\phi(x) G(x,y)\theta[A(y)] d^n x\, d^n y$$

$$+ \int J(x)(-ie_0)\phi^*(x) G(x,y)\theta[A(y)] d^n x\, d^n y\rangle = 0 \tag{C.5}$$

We exhibit the procedure for the first term in $\theta^{(2)}_{\mu\nu}$ viz. $\xi_0 \partial_\mu (\partial . A) A_\nu$. A similar procedure works for other terms. We let $\theta[A(y)] = \partial_\mu \; [A_\nu(y) \;\; \epsilon(y)]$. After integrating by parts and comparing the coefficient of $\epsilon(y)$, we obtain

$$
\begin{aligned}
\langle \xi_0 \partial_\mu (\partial . A) A_\nu(y) \rangle \;=\; & \int J_\rho(x) \; \partial^y_\mu \; \partial^\rho_x \; G(x,y) \; A_\nu(y) \; d^n x \\
& + \int J^*(x) \; ie_0 \phi(x) \; \partial^y_\mu \; G(x,y) \; A_\nu(y) \; d^n x \\
& - \int J(x) \; ie_0 \phi^*(x) \; \partial^y_\mu \; G(x,y) \; A_\nu(y) \; d^n x
\end{aligned}
\tag{C.6}
$$

It is easy to verify now that when the on-shell truncated Green's functions of the right hand side are considered for $q \neq 0$, they vanish either because $\epsilon . k = 0$ for a physical photon or because of the lack of pole at the right position.

Further, following the discussion below Eq. (B.6) of Ref. 14, it is easy to show that $\partial_\mu (\partial . A) A_\nu$ cannot mix with $(\partial_\mu \partial_\nu - \partial^2 \eta_{\mu\nu}) \phi^* \phi$ when one notes the general form of the right hand side of Eq. (B.7) of Ref. 14.

APPENDIX - D

The equation satisfied by $\bar{k}$ is

$$-\lambda \frac{\partial}{\partial\lambda} x^{(1)} - \frac{e}{2}\frac{\partial}{\partial e} x^{(1)} + \left(\lambda \frac{\partial}{\partial\lambda}\bar{k}\right) Z_m^{(1)} + \left(\frac{e}{2}\frac{\partial}{\partial e}\bar{k}\right) Z_m^{(1)} = 0 \tag{D.1}$$

For a given n, we consider the above equation in the following orders:

λ^{n+1} and $\lambda^n e^2$ $\lambda^{n-1}(e^2)^2$ .... $\lambda(e^2)^n$, obtaining a set of equations of constraints on $\bar{k}$. They are 1+n in number, where n runs from 1 to ∞. The number of new $\bar{k}_p$'s that appear in these equations is the same as the number of constraint equations on $\bar{k}$, for a given n. Hence $\bar{k}_p$'s can be fixed uniquely provided a certain condition, which is discussed below, is satisfied.

The equations can be put in the matrix form $A\bar{k} = B$ where A is $(1+n) \times (1+n)$ matrix. The rows of this matrix are labelled below by the orders in which equation (D.1) is considered and columns by the orders in which $\bar{k}$ is considered in these equations. The matrix can be so arranged that order of λ is decreasing across the row and down the column. We know that in each order $\lambda^r(e^2)^q$ (r and q run from 0 to ∞) in coupling constants, there is a new $\bar{k}_p$ that appears. Taking into consideration all the above mentioned facts, it is possible to show that the matrix A has the triangular form. The unique solution of the set of equations exists if det $A \neq 0$.

For a given n, A has the form given below:

$$A = \begin{array}{l} \\ \lambda^{n+1} \\ \lambda^{n}\, e^{2} \\ \lambda^{n-1}(e^{2})^{2} \\ \vdots \\ \lambda(e^{2})^{n} \end{array} \begin{array}{c} \begin{array}{cccc} \lambda^{n} & \lambda^{n-1}e^{2} & \lambda^{n-2}(e^{2})^{2} & \cdots \end{array} \\ \begin{bmatrix} a_{11} & & 0 & \\ \cdots & a_{22} & & \\ \cdots & \cdots & a_{33} & \\ & & & \ddots \\ & & & \end{bmatrix} \end{array}$$

where $a_{11} = a_{22} = a_{33} = \ldots.$

Looking at equation (D.1), we find that the diagonal elements of the matrix A are all equal. They are the simple pole term in $O(\lambda)$ in Z_m, which we know is nonzero. It ensures that det $A \neq 0$.

APPENDIX - E.1

In section 4.3 we used the result that

$$\text{p.p}\left[G\, Z_m^{-1} + \frac{(n-2)}{2(1-n)}\, Z_m^{-1}\right] = 0 \qquad \text{(E.1.1)}$$

upto $0(\lambda^3,\ \lambda g'^2,\ g'^4,\ \lambda g^2,\ g^4,\ g^2 g'^2)$

To prove the above relation, we consider the renormalization of $\theta_\mu^{\mu\,\mathrm{imp}}$ where $\theta_{\mu\nu}^{\mathrm{imp}} = \theta_{\mu\nu} + \frac{(n-2)}{2(1-n)}\,(\partial_\mu\partial_\nu - \eta_{\mu\nu}\partial^2)\,(\phi^\dagger\phi)$.

$$\theta_\mu^{\mu\,\mathrm{imp}} = (n-4)\left[\frac{1}{4}F_{\mu\nu}^i F^{\mu\nu i} + \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \lambda_o(\phi^+\phi)^2 + \frac{\alpha_o}{2}(\partial.B)^2 + \frac{\xi_o}{2}(\partial.A^a)^2 - \frac{i}{2}\bar\psi_i\gamma^\mu(\overrightarrow{\partial}_\mu - ig_o\frac{\tau}{2}.A_\mu - \frac{ig_o'}{2}YB_\mu)\psi_i + \frac{i}{2}\bar\psi_i(\overleftarrow{\partial}_\mu + ig_o\frac{\tau}{2}.A_\mu + ig_o'\frac{Y}{2}B_\mu)\gamma^\mu\psi_i\right]$$

$$- \frac{(n-2)}{2}\left[\phi^\dagger\frac{\delta S}{\delta\phi^\dagger} + \frac{\delta S}{\delta\phi}\phi\right] + 2m_o^2\,\phi^\dagger\phi - \frac{3}{2}\left[\bar\psi\frac{\delta S}{\delta\bar\psi} - \frac{\delta S}{\delta\psi}\psi\right]$$

$$- (n-2)\partial^\mu\left[\xi_o(\partial.A^a)A_\mu^a - \bar C_a\tilde D_\mu^{ab}C_b\right] - (n-2)\,\alpha_o\left[\partial^\rho(B_\rho\,\partial.B)\right]$$

$$- (n-2)\,\bar C\frac{\delta S}{\delta\bar C} \qquad \text{(E.1.2)}$$

All the terms except the one in the first square bracket are finite [14,15].

Hence

$$\langle\theta_\mu^{\mu\,\mathrm{imp}}\rangle = \text{finite} + (n-4)\left\langle\left[-\frac{1}{4}F_{\mu\nu}^i F^{\mu\nu i} + \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \lambda_o(\phi^+\phi)^2\right.\right.$$

$$+ \frac{\xi_o}{2}(\partial.A^a)^2 + \frac{\alpha_o}{2}(\partial.B)^2 - \frac{i}{2}\bar\psi_i\gamma^\mu(\overrightarrow{\partial}_\mu - ig_o\frac{\tau}{2}.A_\mu - \frac{iYg_o'B_\mu}{2})\psi_i +$$

$$+ \frac{1}{2}\bar{\psi}_1(\overleftarrow{\partial}_\mu + ig_o\frac{\tau.A_\mu}{2} + ig'_o\frac{Y}{2}B_\mu)\gamma^\mu\psi_1\Big]>$$

We shall consider the renormalization of the operator in the square bracket written above.

The following set of unrenormalized operators is closed under renormalization :

$$O_1 = \frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i} + \frac{1}{4}G_{\mu\nu}G^{\mu\nu} - \lambda_o(\phi^+\phi)^2 + \frac{\xi_o}{2}(\partial.A^a)^2 + \frac{\alpha_o}{2}(\partial.B)^2$$

$$-\frac{i}{2}\bar{\psi}_1\gamma^\mu(\overrightarrow{\partial}_\mu - ig_o\frac{\tau.A_\mu}{2} - \frac{iYg'_o B_\mu}{2})\psi_1 + \frac{i}{2}\bar{\psi}_1(\overleftarrow{\partial}_\mu + ig_o\frac{\tau.A_\mu}{2} + ig'_o\frac{Y}{2}B_\mu)\gamma^\mu\psi_i$$

$$O_2 = m_o^2\,\phi^\dagger\phi \quad , \qquad O_3 = \phi^\dagger\frac{\delta S}{\delta\phi^\dagger} + \frac{\delta S}{\delta\phi}\phi$$

$$O_4 = \frac{1}{2}\left[\bar{\psi}\frac{\delta S}{\delta\bar{\psi}} - \frac{\delta S}{\delta\psi}\psi\right] , \qquad O_5 = A^a_\mu\frac{\delta S}{\delta A^a_\mu} + \partial_\mu\bar{C}^a\tilde{D}^{\mu ab}C_b$$

(where $\tilde{S} = S$ + g.f term)

$$O_6 = \frac{\delta S}{\delta C^a}C^a \quad , \qquad O_7 = B_\mu\frac{\delta S}{\delta B_\mu}$$

$$O_8 = \frac{1}{4}F^i_{\mu\nu}F^{\mu\nu i} + \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + \frac{\xi_o}{2}(\partial.A^a)^2 + \frac{\alpha_o}{2}(\partial.B)^2 - \frac{i}{2}\bar{\psi}\gamma^\mu$$

$$\times(\overrightarrow{\partial}_\mu - ig_o\frac{\tau.A_\mu}{2} - \frac{ig'_o}{2}YB_\mu)\psi_i + \frac{i}{2}\bar{\psi}_i(\overleftarrow{\partial}_\mu + ig_o\frac{\tau.A_\mu}{2} + ig'_o\frac{Y}{2}B_\mu)\gamma^\mu\psi_i$$

$$O_9 = \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} \quad , \qquad O_{10} = \frac{\xi_o}{2}(\partial.A^a) + \frac{\alpha_o}{2}(\partial.B)^2$$

$$O_{11} = \partial^2(\phi^\dagger\phi)$$

$$\left\{ O_i \right\}^{UR} = Z_{ij}\left\{ O_j \right\}^{R} :$$

where the renormalization matrix Z_{ij} is given below:

$$\begin{bmatrix} Z_{11} & Z_{12} & Z_{13} & Z_{14} & Z_{15} & Z_{16} & Z_{17} & Z_{18} & Z_{19} & Z_{110} & Z_{111} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ Z_{81} & Z_{82} & Z_{83} & Z_{84} & Z_{85} & Z_{86} & Z_{87} & Z_{88} & Z_{89} & Z_{810} & Z_{811} \\ Z_{91} & Z_{92} & Z_{93} & Z_{94} & Z_{95} & Z_{96} & Z_{97} & Z_{98} & Z_{99} & Z_{910} & Z_{911} \\ 0 & 0 & Z_{103} & Z_{104} & Z_{105} & Z_{106} & Z_{107} & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & Z_m^{-1} \end{bmatrix}$$

The renormalization constants Z_{11} to $Z_{1\,10}$ are simple poles. [14,15] Hence ensuring the finiteness of $\theta_{\mu}^{\mu\,imp}$ at $q = 0$. We are interested in determining whether $Z_{1\,11}$ has double poles

in $O(\lambda^3, \lambda g^2, \lambda g'^2, g^4, g'^4, g^2 g'^2)$. To determine it, we will consider the RG equation for $Z_{1\,11}$ which is given by

$$\gamma_{1\,11} = Z_{11}^{-1}\, \mu \frac{\partial}{\partial\mu} Z_{1\,11} + Z_{18}^{-1}\, \mu \frac{\partial}{\partial\mu} Z_{8\,11} + Z_{19}^{-1}\, \mu \frac{\partial}{\partial\mu} Z_{9\,11}$$

$$+ Z_{1\,11}^{-1}\, \mu \frac{\partial}{\partial\mu} Z_m^{-1} \qquad \text{(E.1.3)}$$

Consider the above equation in $O\left(\frac{g^2 g'^2}{\epsilon}\right)$. We know that

i) $Z_{1\,11}$ is zero in $O(g^2)$, $O(g'^2)$ and $O(\lambda^2)$

ii) $Z_{8\,11}$, $Z_{9\,11}$ are zero in $O(g^2)$ and $O(g'^2)$. Also we notice that the simple pole terms in $Z_{1\,11}^{-1}$ are the simple pole term of $Z_{1\,11}$. (It can be seen by writing down the expression for $Z_{1\,11}^{-1}$ explicitly). All the above mentioned facts lead to the conclusion that $Z_{1\,11}$ does not have double poles in $O(g^2 g'^2)$.

In the context of scalar electrodynamics and non abelian gauge theory with scalars, it has been shown that the renormalization constant for the mixing of operator O_1 with $\partial^2(\phi^*\phi)$ does not have double poles in $O(\lambda^3, \lambda g'^2, \lambda g'^4)$ and $O(\lambda^3, \lambda g^2, g^4)$ respectively[14,15] where $O_1 = -\frac{\lambda_o}{4!}(\phi^*\phi)^2 + \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\alpha_o}{2}(\partial.A)^2$ for scalar QED and

$$O_1 = -\frac{\lambda_o}{4!}(\phi^T\phi)^2 + \frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{\xi_o}{2}\sum_a(\partial.A^a)^2$$ for non abelian theory with scalars.

Hence we conclude that $Z_{1\ 11}$ in W-S model will not have double poles in $O(\lambda^3, \lambda g'^2, \lambda g'^4, g^4, \lambda g^2, g^2 g'^2)$.

APPENDIX - E.2

To show that $\bar{k}_n(\lambda, g^2, g'^2, \alpha, \xi)$ is independent of the gauge parameters, we consider

$$\theta_\mu^{\mu\,imp} = \theta_\mu^\mu - \left[G(\lambda, g^2, g'^2, \xi, \alpha) + \bar{k}(\lambda, g^2, g'^2\, \frac{m^2}{\mu^2}, \alpha, \xi) Z_m\right] (1-n)\partial^2(\phi^\dagger\phi)$$

Where θ_μ^μ is the trace of $\theta_{\mu\nu}$ given by equation (6). Since the physical matrix elements of $\theta_{\mu\nu}^{imp}$ should be gauge independent, it implies that the physical matrix elements of $\theta_\mu^{\mu\,imp}$ should be gauge independent too. It is easy to see that $\theta_\mu^{\mu\,imp}$ can be split up into four parts $\theta_\mu^{\mu(1)}$, $\theta_\mu^{\mu(2)}$, $\theta_\mu^{\mu(3)}$ and $\theta_\mu^{\mu(4)}$ where $\theta_\mu^{\mu(1)}$ is a gauge invariant operator, hence its matrix elements (barring some exceptional momenta) are independent of gauge parameters. $\theta_\mu^{\mu(2)}$ contain the operators which are all dependent on the gauge parameters. They do not mix with $\partial^2(\phi^\dagger\phi)$ [14,15].

$$\theta_\mu^{\mu(3)} = - G(\lambda, g^2, g'^2\, \xi, \alpha)\ (1-n)\partial^2(\phi^\dagger\phi) \text{ and}$$

$$\theta_\mu^{\mu(4)} = - \bar{k}(\lambda, g^2, g'^2\, \frac{m^2}{\mu^2}, \alpha, \xi)\ \partial^2(\phi^\dagger\phi)^R$$

The arguments given in the previous paragraph imply that

$$\left\{\langle\theta_\mu^\mu\rangle\right\}^{div} = G'(\lambda, g^2, g'^2, \xi, \alpha, \epsilon)\ \partial^2\ \langle\phi^\dagger\phi\rangle^R$$

$$= \left\{\langle\theta_\mu^{\mu(1)}\rangle\right\}^{div} \qquad \text{where } G' = G\, Z_m$$

From the above expression it follows that $G'(\lambda, g^2, g'^2\ \xi, \alpha, \epsilon)$ is gauge independent, implying G is gauge independent. Hence $\theta_\mu^{\mu(3)}$ is gauge independent operator with gauge independent coefficient. It leads us to the conclusion that if $\theta_\mu^{\mu\, imp}$ is to have gauge independent matrix elements, $\bar{k}$ should be independent of the gauge parameters.

APPENDIX - F

The equation satisfied by $\bar{k}$ is

$$-\lambda \frac{\partial}{\partial\lambda} X^{(1)} - \frac{g}{2} \frac{\partial}{\partial g} X^{(1)} - \frac{g'}{2} \frac{\partial}{\partial g'} X^{(1)} + (\lambda \frac{\lambda}{\partial\lambda} \bar{k}) Z_m^{(1)}$$

$$+ (\frac{g}{2} \frac{\partial}{\partial g} \bar{k}) Z_m^{(1)} + (\frac{g'}{2} \frac{\partial}{\partial g'} \bar{k}) Z_m^{(1)} = 0. \qquad \text{(F.1)}$$

For a given n, we consider the above equation in the following orders:

λ^{n+1}

$\lambda^n g^2 \quad \lambda^{n-1} (g^2)^2 \; . \; . \; . \; . \; . \; \lambda (g^2)^n$

$\lambda^n g'^2 \quad \lambda^{n-1} (g'^2)^2 \; . \; . \; . \; . \; . \; \lambda (g'^2)^n$

$\lambda (g'^2) g^2 \quad \lambda (g'^2)^{n-2} (g^2)^2 \; . \; . \; . \; . \; . \; \lambda (g'^2)(g^2)^{n-1}$

$\lambda^2 (g'^2)^{n-2} g^2 \quad \lambda^2 (g'^2)^{n-3} (g^2)^2 \; . \; . \; . \; . \; \lambda^2 (g'^2)(g^2)^{n-1}$

$\lambda^3 (g'^2)^{n-3} g^2 \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; \lambda^3 (g'^2)(g^2)^{n-3}$

.
.

$\lambda^{n-1} \; g'^2 \; g^2$

[The orders are $1+2n+\sum_{m=1}^{n-1} n-m = p$ in number (n running from 1 to ∞)]

Thus we obtain a set of equations of constraints on $\bar{k}$. The number of new $\bar{k}_j$'s that appear in this set of equations is the same as

the number of constraint equations on $\bar{k}$, to be considered for a given n. Hence it is possible to choose $\bar{k}_j$'s uniquely provided a condition discussed below, is satisfied.

The set of equations can be put in the matrix from $A\bar{k}=B$ where A is pxp matrix. The rows of this matrix A are labelled below by the orders in which the equation (F.1) is considered and the columns by the order in which $\bar{k}$ is considered in this equation. The rows and columns of matrix A are so arranged that order of λ is decreasing across the row and down the column. We see that in each order $(\lambda^r (g'^2)^q (g^2)^s$, r,q,s running from 0 to ∞) in coupling constants, there is new $\bar{k}_j$ that appears. Thus it is possible to put the matrix A in the triangular form given below.

For a given n :

$$
\begin{array}{c|cccccc}
 & \lambda^n & \lambda^{n-1}g^2 & \lambda^{n-1}g'^2 & \lambda^{n-2}(g^2)^2 & \lambda^{n-2}(g'^2)^2 & \lambda^{n-2}g^2g'^2 \ldots \\
\hline
\lambda^{n+1} & a_{11} & & & & & \\
\lambda^n(g^2) & \ldots & a_{22} & & 0 & & \\
A = \lambda^n(g'^2) & \ldots & \ldots & a_{33} & & & \\
\lambda^{n-1}(g^2)^2 & & & & \cdot & & \\
\lambda^{n-1}(g'^2)^2 & & & & & \cdot & \\
\lambda^{n-1}\, g^2g'^2 & & & & & & \\
\vdots & & & & & & \\
\lambda(g^2)^n & & & & & & \\
\lambda(g'^2)^n & \ldots\ldots & \ldots\ldots & & \ldots\ldots & & \ldots\ldots
\end{array}
$$

It is easy to see that

$a_{11} = a_{22} = a_{33} = \ldots\ldots\ldots\ldots\ldots$, the diagonal elements are all equal and they are the simple pole term in $O(\lambda)$ in Z_m which we know is non zero. Hence we conclude that det $A \neq 0$, ensuring that the unique solution to the set of equations, we considered, exists.

APPENDIX-G

In this appendix, we wish to first consider as to what is the appropriate action to be taken in a finite volume and then discuss the possibility of letting quantization volume $V \to \infty$.

We shall first see as to how to define a finite action for the interaction of gravity with matter to first order in $h_{\mu\nu}$. At first let the system be closed in a finite 4-dimensional box. Let q denote an allowed four momentum. Let $\langle\tilde{\theta}^{imp}_{\mu\nu}(q)\rangle$ denote renormalized a matrix element of $\theta^{imp}_{\mu\nu}$ at momentum q. we define

$$\langle S_1 \rangle = \frac{1}{2} \sum_q h_{\mu\nu}(q) \langle \tilde{\theta}^{imp}_{\mu\nu}(q) \rangle \tag{G.1}$$

where we have defined

$$h_{\mu\nu}(q) = \int h_{\mu\nu}(x)\, e^{iq.x}\, d^4x \tag{G.2}$$

We define

$$\langle \theta^{\mu\nu\, imp}(x) \rangle \equiv \sum_q \tilde{\theta}^{imp}_{\mu\nu}(q)\, e^{iq.x} \tag{G.3}$$

Then (G.1) becomes

$$S_1 = \int h_{\mu\nu}(x)\, \theta^{\mu\nu\, imp}(x)\, d^4x \tag{G.4}$$

By construction S_1 is finite. The renormalized Green's functions of the theory to first order in h are obtained from the action

$$S = S_o + S_1 \tag{G.5}$$

with

$$S_o = \int_V d^4x \, [\tfrac{1}{2} \partial_\mu \phi \, \partial^\mu \phi \; - \tfrac{1}{2} m_o^2 \phi^2 - \frac{\lambda_o}{4!} \phi^4] \qquad (G.6)$$

An important point to notice is that this action of (G.5) differs from the infinite space action normally chosen:

$$S' = \int d^4x \sqrt{-g} \, [\tfrac{1}{2} \partial_\mu \phi \, \partial^\mu \phi \; - \tfrac{1}{2} m_o^2 \phi^2 - \frac{\lambda_o}{4!} \phi^4]$$

$$+ \tfrac{1}{2} H_o(\epsilon) \int d^4x \sqrt{-g} \, R \, \phi^2 \qquad (G.7)$$

(evaluated to first order in h) when it is applied to a finite volume. This action is

$$S'' = S_o + S_1' \qquad (G.8)$$

with

$$S_1' = \tfrac{1}{2} \int h_{\mu\nu} \, \theta^{c\mu\nu} \, d^4x + \tfrac{1}{2} H_o(\epsilon) \int_V (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial^2) h_{\mu\nu} \, \phi^2 d^4x$$

$$(G.9)$$

Now, the differences $S\text{-}S'' = S_1 - S_1'$ is

$$S_1 - S_1' = \tfrac{1}{2} H_o(\epsilon) \, [\int_V h_{\mu\nu} \, (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial^2) \, \phi^2 d^4x$$

$$- \int_V (\partial^\mu \partial^\nu - \eta^{\mu\nu} \partial^2) \, h_{\mu\nu} \, \phi^2 \, d^4x \,]$$

$$= \tfrac{1}{2} H_o(\epsilon) \int_S d\Sigma_\mu \, (\tilde{h}^{\mu\nu} \, \partial_\nu \, \phi^2 - \partial_\nu \, \tilde{h}^{\mu\nu} \, \phi^2)$$

$$(\tilde{h}^{\mu\nu} = h^{\mu\nu} - \eta^{\mu\nu} \, h^\alpha_\alpha) \qquad (G.10)$$

is a surface term in 4-dimensions. We propose that even in the infinite volume case, the surface term should be included and limit $V \longrightarrow \infty$ of this should be found out, which <u>may</u> not vanish. In an example dealt with in this appendix, this is so, as seen below. The surface term is a divergent term (as $\epsilon \longrightarrow 0$) indicating that S" cannot be used to obtain finite matrix elements.

It should, thus, be noted that while the second term on the right hand side of Eq. (G.7) may vanish if R = 0, it is not S" but S which is the correct action leading to <u>finite</u> result.

Throughout the discussion of the text, we have assumed that the system is enclosed in a finite volume V. It may appear that a result such as that of Eq. (6.21) would vanish if $V \to \infty$. We shall show with a concrete example of a wavepacket that in the latter case ($V \to \infty$), the results are still unaffected and the $\frac{1}{V}$ in the Eq. (6.21) is replaced simply by $\frac{1}{V_{eff}}$ where V_{eff} is of the order of the volume over which the particle wavefunction is spread. Thus, this example will show that finiteness of the quantization volume is by no means a necessity for a nontrivial result. We could exemplify this with the help of the process of scattering of two scalars discussed in the text, but the demonstration becomes much simplified if we, instead, consider particle production process $\phi \longrightarrow 3\phi$ in presence of a black hole or a neutron star and we shall adopt this example for our purpose.

We shall let the initial particle wavefunction $(2p_oV)^{-1/2}$ exp $(ip.x)$ to be replaced by a wave packet

$\int \frac{d^3k\ a(\vec{k})}{\sqrt{2k_o}} e^{ik.x}$ [The choice $a(\vec{k}) = \frac{1}{\sqrt{V}}\ \delta^{(3)}(\vec{k}-\vec{p})$ would restore this wavepacket to the plane wave]. As for the final particles we shall keep them to be plane waves. In any case, we shall note that the 1/V factor coming from square of each final state wave-function is exactly cancelled by $\frac{V}{(2\pi)^3}$ contained in the number of states of the final state particle in the expression for decay width. Thus the resultant factor $\frac{1}{V}$ may be thought of as arising from the initial state plane wave. It is this factor that we wish to show is generally replaced by $\frac{1}{V_{eff}}$. To this end we shall allow an $a(\vec{k})$ of the form of a gaussian

$$a(\vec{k}) = A\ \sigma^{-3/2}\ e^{-(\vec{k}-\vec{p})^2/\sigma^2} \qquad (G.11)$$

We shall let σ very small so that the spread of $a(\vec{k})$ around $\vec{p}$ is small. [Where A is a dimensionless normalization constant]. Let us denote by q_1, q_2, q_3 the four momenta of the final state particles. Let $\vec{q} = \vec{q}_1 + \vec{q}_2 + \vec{q}_3 - \vec{p}$ be the (three-vector) momentum injected by gravity. Let $Q_o = q_{10} + q_{20} + q_{30}$. Then a straightforward calculation shows that in the case of an initial plane wave, there is a factor

$$\frac{1}{\sqrt{2p_oV}}\ \delta(p_o-Q_o)\ (\cos q\, r_1 - \cos q\, r_2)\ G(p,\ q_1, q_2, q_3) \qquad (G.12)$$

in the amplitude which, in the case of an initial wave packet of Eq. (G.11), is replaced by

$$\int \frac{d^3k\ a(\vec{k})}{\sqrt{2k_o}} \delta(k^o-Q^o) \left\{\cos\, |Q_o\hat{n} - \vec{Q}|r_1 - \cos|Q_o\hat{n} - \vec{Q}|r_2\right\}$$

$$\times\ G(\ k,\ q_1,\ q_2,\ q_3\ ) \qquad (G.13)$$

Here, $\hat{n} = \vec{k}/|\vec{k}|$. As σ is very small, so that spread of values of $\vec{k}$ around $\vec{p}$ is small, we may replace $G(k, q_1, q_2\ q_3)$ by $G(p, q_1, q_2, q_3)$ in (G.13) assuming that the former is a slowly varying function of $\vec{k}$ as compared to the cosine functions. Thus, we need to compare a factor

$$\frac{1}{\sqrt{2p_0 V}} \quad \delta(p_0 - Q_0) \cos q\ r \tag{G.14}$$

with

$$\int \frac{d^3k\ a(\vec{k})}{\sqrt{2k_0}} \quad \delta(k_0 - Q_0) \cos |\ Q_0\hat{n} - \vec{Q}\ |\ r \tag{G.15}$$

where r is either r_1 or r_2

We shall let Q (fixed by final external momenta) to be along z-axis. We shall present the calculation for (G.15) in the relatively simple case in which $\vec{p}$ is along the x axis. (The general case when $\vec{p}$ is arbitrarily oriented gives similar results).

We can then express, for small σ,

$$a(k) \simeq A\ \sigma^{-3/2} \quad e^{-(k-p)^2/\sigma^2} \quad e^{-kp\phi^2/\sigma^2} \quad e^{-kp\tilde{\theta}^2/\sigma^2}$$

(G.16)

with $\tilde{\theta} = \frac{\pi}{2} - \theta$, where we have used the approximation in which ϕ and $\tilde{\theta}$ are small, as the wavepacket is sharply centered around $\vec{k} = \vec{p}$. In a similar manner we can expand, for $\tilde{\theta}$ small,

$$\cos|Q_o\hat{n} - \vec{Q}|r \simeq \cos\eta r \cos\xi r\tilde{\theta} + \sin\eta r \sin\xi r\tilde{\theta}$$

$$\eta \equiv (Q_o^2 + Q^2)^{1/2} \qquad \xi = QQ_o/(Q^2 + Q_o^2)$$

The second term on the right hand side of (G.17) does not contribute as it is an odd function of $\tilde{\theta}$. A straightforward calculation then yields that the leading contribution to (G.15) is

$$= \frac{A\,\Pi^{3/2}\,\sigma^{3/2}}{\sqrt{2p}} \cos\eta r \quad e^{-\xi^2 r^2 \sigma^2/4p^2} \left[\frac{1}{\sqrt{\pi\sigma}}\; e^{-(Q_o - p_o)^2/\sigma^2}\right] \tag{G.18}$$

Consider first $r = r_1 =$ finite. Expression (G.18) is to be compared with (G.14). Noting that $q = \eta$ in the present case (when $\sigma=0$) and $\exp\left[-\xi^2 r^2 \sigma^2/4p^2\right] \longrightarrow 1$ as $\sigma \ll 1/r_1$; $[r_1 =$ finite$]$ and the expression in the square bracket of (G.18) approaches $\delta(Q_o - p_o)$ as $\sigma \longrightarrow 0$, we see that the factor $1/\sqrt{V}$ is replaced by

$$A\,\Pi^{3/2}\,\sigma^{3/2} \equiv 1/\sqrt{V_{eff}} \tag{G.19}$$

in (G.18). Noting that $\sigma \sim$ order of spread k_x, the order of spread of the wavepacket in coordinate space is $1/\sigma$ per dimension. Thus σ^{-3}, indeed represents the volume over which the wavepacket is spread. This is proportional to V_{eff}. [This discussion is valid provided the spread of the packet in coordinate space is very large (as compared r_1)].

On the other hand when $r = r_2$, $\exp\left[\frac{-\xi^2 r_2^2\,\sigma^2}{2}\right] \longrightarrow 0$

as $r_2 \longrightarrow \infty$ for a fixed σ. Thus the factor (G.12) is replaced by

$$\frac{1}{\sqrt{2p_o\, V_{eff}}}\ \delta(p_o - Q_o)\cos q\, r_1 \tag{G.20}$$

yielding about the same numerical results.

The investigation as to what happens as $r_1 \longrightarrow 0$. cannot be done for a black hole or a neutron star because h becomes large for $r < r_1$, and our approximation breaks down. In the case of say earth, one needs a solution anologous to Schwarzschild solution, inside the earth and that can be matched with the Schwarzschild solution at the boundary.

APPENDIX-H

In this appendix, we shall show that the term $-\frac{8\pi r_o}{|\bar{q}|^2}\int_{x_1}^{x_2}\frac{\cos qr_o x}{x-1}dx$ in Eq. (6.34) is of $O(\frac{1}{|\bar{q}|^3})$ for $qr_o \gg 1$. To this end, consider the integral

$$\int_{x_1}^{x_2}\frac{\cos qr_o x}{x-1}\,dx \tag{H.1}$$

For a fixed qr_o, let $y_1 \geq x_1$ be the zero of $\cos(q\,r_o x)$ nearest to $x = x_1$ and let $y_2 \leq x_2$ be the zero of $\cos(qr_o x)$ nearest to x_2 which satisfies $y_2 - y_1 = \frac{2N\pi}{qr_o}$ where N an integer. We divide the internal (y_1, y_2) into 2N equal intervals: (y_1, ξ_1), (ξ_1, ξ_2), ...(ξ_{2N-1}, y_2) where $\xi_j \equiv y_1 + \frac{j\pi}{qr_o}$. For concreteness, let $\cos(qr_o x)$ be positive in (y_1, ξ_1). [An identical result holds if it is negative there.] We write

$$\int_{x_1}^{x_2}\frac{\cos qr_o x}{x-1}dx = \int_{x_1}^{y_1}\frac{\cos(qr_o x)}{x-1}dx + \int_{y_2}^{x_2}\frac{\cos(qr_o x)}{x-1}dx$$

$$+\sum_{n=0}^{N-1}\left\{\int_{\xi_{2n}}^{\xi_{2n+1}}\frac{\cos(qr_o x)}{x-1}dx + \int_{\xi_{2n+1}}^{\xi_{2n+2}}\frac{\cos(qr_o x)}{x-1}dx\right\} \tag{H.2}$$

Now $y_1 - x_1 < \frac{\pi}{qr_o}$ and $x_2 - y_2 < \frac{2\pi}{qr_o}$. Thus,

$$\left|\int_{x_1}^{y_1}\frac{\cos qr_o x}{x-1}dx\right| \leq \frac{1}{x_1-1}\;\frac{\pi}{qr_o} = 0\left(\frac{1}{qr_o}\right)$$

$$\left| \int_{y_2}^{x_2} \frac{\cos qr_o x}{x-1}\, dx \right| \leq \frac{1}{y_2 - 1} \frac{2\pi}{qr_o} = 0\left(\frac{1}{qr_o}\right) \tag{H.3}$$

For $qr_o >> 1$, $\frac{1}{x-1}$ does not vary much in the interval (ξ_{2n}, ξ_{2n+1}). And thus we may approximate

$$\int_{\xi_{2n}}^{\xi_{2n+1}} \frac{\cos(qr_o x)}{x-1}\, dx + \int_{\xi_{2n+1}}^{\xi_{2n+2}} \frac{\cos(qr_o x)}{x-1}\, dx = \frac{2/qr_o}{\xi_{2n-1}} - \frac{2/qr_o}{\xi_{2n+1}-1}$$

Thus the last term in Eq. (H.2) can be approximated by (approximation getting better and better as $qr_o \to \infty$)

$$\frac{2}{qr_o} \sum_{n=0}^{N-1} \left(\frac{1}{\xi_{2n}-1} - \frac{1}{\xi_{2n+1}-1}\right)$$

$$= \frac{2}{qr_o} \sum_{n=0}^{N-1} \frac{\pi/qr_o}{(\xi_{2n}-1)(\xi_{2n+1}-1)}$$

$$\simeq \frac{2}{qr_o} \int_{y_1}^{y_2} \frac{dx}{(x-1)^2} = 0\left(\frac{1}{qr_o}\right) \tag{H.4}$$

Using Eqn. (H.3) and (H.4), one obtains

$$\frac{8\,\Pi r_o}{|\vec{q}|^2} \int_{x_1}^{x_2} \frac{\cos q\, r_o x}{(x-1)}\, dx = 0\left(\frac{1}{\vec{q}^{\,3}}\right)$$

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